

# From Generalized Dirac Equations to a Candidate for Dark Energy

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We consider extensions of the Dirac equation with mass terms  $m_1 + i\gamma^5 m_2$  and  $i m_1 + \gamma^5 m_2$ . The corresponding Hamiltonians are Hermitian and pseudo-Hermitian (“ $\gamma^5$  Hermitian”), respectively. The fundamental spinor solutions for all generalized Dirac equations are found in the helicity basis and brought into concise analytic form. We postulate that the time-ordered product of field operators should yield the Feynman propagator ( $i\epsilon$  prescription), and we also postulate that the tardyonic as well as tachyonic Dirac equations should have a smooth massless limit. These postulates lead to sum rules that connect the form of the fundamental field anticommutators with the tensor sums of the fundamental plane-wave eigenspinors and the projectors over positive-energy and negative-energy states. In the massless case, the sum rules are fulfilled by two egregiously simple, distinguished functional forms. The first sum rule remains valid in the case of a tardyonic theory and leads to the canonical massive Dirac field. The second sum rule is valid for a tachyonic mass term and leads to a natural suppression of the right-handed helicity states for tachyonic particles, and left-handed helicity states for tachyonic spin-1/2 antiparticles. When applied to neutrinos, the theory contains a free tachyonic mass parameter. Tachyons are known to be repulsed by gravity. We discuss a possible role of a tachyonic neutrino as a contribution to the accelerated expansion of the Universe (“dark energy”).

PACS numbers: 11.10.-z, 03.70.+k, 95.85.Ry, 95.36.+x, 98.80.-k

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## I. INTRODUCTION

### A. Generalized Dirac Equations: Mass Terms and Dispersion Relations

Dirac is often quoted as saying in some of his talks that the equation that carries his name [1, 2] is “more intelligent than its inventor”. Of course, it needs to be added that it was Dirac himself who found most of the additional insight. Here, we are concerned with extensions of the Dirac equation which contain both tardyonic and tachyonic mass terms. Tardyonic (subluminal) mass terms lead to dispersion relations of the form  $E = \sqrt{\vec{p}^2 + m^2}$ , whereas tachyonic mass terms lead to superluminal dispersion relations of the form  $E = \sqrt{\vec{p}^2 - m^2}$ , where  $E$  is the energy and  $\vec{p}$  is the momentum. The generalized, matrix-valued mass term  $M$  enters the Dirac equation in the form  $(i\gamma^\mu \partial_\mu - M)\psi(x) = 0$ . The  $\gamma^\mu$  are  $4 \times 4$  matrices that fulfill the relations  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  where we choose the space-time metric as  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . The  $\partial_\mu$  denote the partial derivative  $\partial/\partial x^\mu$  with respect to the space-time coordinate  $x^\mu = (t, \vec{x})$ . It is quite surprising that a systematic presentation of the solutions of the generalized Dirac equations  $(i\gamma^\mu \partial_\mu - M)\psi(x) = 0$ , in the helicity basis [3], has not been recorded in the literature to the best of our knowledge. While the following discussion is somewhat technical, we believe that it will be beneficial to give their explicit form, in order to fix ideas for the following discussion.

We set  $\hbar = c = \epsilon_0 = 1$  and use the Dirac matrices in the standard representation

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad (1.1)$$

and define  $\vec{\alpha} = \gamma^0 \vec{\gamma}$ . For the ordinary Dirac theory, one has  $M = m_1$  (one should say more precisely  $M = m_1 \mathbb{1}_{4 \times 4}$ ) with a real mass  $m_1$ ,

$$(i\gamma^\mu \partial_\mu - m_1) \psi(x) = 0. \quad (1.2)$$

The dispersion relation is  $E = \sqrt{\vec{p}^2 + m_1^2}$ . The corresponding Dirac Hamiltonian reads

$$H^{(1)} = \vec{\alpha} \cdot \vec{p} + \beta m_1. \quad (1.3)$$

Extensions of the Dirac equation with pseudoscalar mass terms that contain the fifth current have been introduced in the literature. In Ref. [4], it is shown that for a mass term of the form  $M = m_1 + i\gamma^5 m_2$ , the fermion propagator may obtain nontrivial gradient corrections already at the first order in derivative expansion, for a position-dependent mass. In that case, the fermion self-energy may contribute to a conceivable explanation for  $\mathcal{CP}$ -violation during electroweak baryogenesis, as pointed out in Ref. [4]. We thus study the following, generalized form of the tardyonic (subluminal) Dirac equation,

$$(i\gamma^\mu \partial_\mu - m_1 - i\gamma^5 m_2) \psi(x) = 0. \quad (1.4)$$

The dispersion relation is  $E = \sqrt{\vec{p}^2 + m_1^2 + m_2^2}$ . The Hermitian tardyonic Hamiltonian operator reads as

$$H^{(t)} = \vec{\alpha} \cdot \vec{p} + \beta m_1 + i\beta \gamma^5 m_2. \quad (1.5)$$

We may indicate a further motivation for our study; namely, the unitarity of the  $S$  matrix implies the existence of useful relations [5] for the even powers  $(m_2)^{2n}$  obtained upon expanding a one-loop amplitude, formulated with a mass term  $m_1 + i\gamma^5 m_2$ , in powers of  $m_2$ . This implies that a better understanding of the tardyonic equation with two mass terms could be of much more general interest.

It has not escaped our attention that the chiral transformation

$$\mu \exp(i\gamma^5 \theta) = \mu \cos \theta + i\gamma^5 \mu \sin \theta = m_1 + i\gamma^5 m_2 \quad (1.6)$$

connects the two Hamiltonians  $H^{(1)}$  and  $H^{(t)}$  for  $m_1 = \mu \cos \theta$  and  $m_2 = \mu \sin \theta$ , but it is computationally easier and more instructive to consider the real and imaginary parts of the mass term separately.

Within a systematic approach to generalized Dirac equations with pseudoscalar mass terms, we also consider tachyonic (superluminal) mass terms of the form  $M = \gamma^5 m$  which induce a superluminal dispersion relation  $E = \sqrt{\vec{p}^2 - m^2}$ . The corresponding generalized Dirac equation has been named the “tachyonic Dirac equation” and reads as follow [6–10],

$$(i\gamma^\mu \partial_\mu - \gamma^5 m) \psi(x) = 0. \quad (1.7)$$

The corresponding Hamiltonian reads

$$H_5 = \vec{\alpha} \cdot \vec{p} + \beta \gamma^5 m. \quad (1.8)$$

The relation  $H_5 = \gamma^0 H_5^+(-\vec{r}) \gamma^0$  has been given in Refs. [9, 10]. However, it is much more instructive to observe that  $H_5$  is  $\gamma^5$  Hermitian, i.e.,  $H_5 = \gamma^5 H_5^+ \gamma^5$ . The concept of  $\gamma^5$  Hermiticity is known in lattice theory [11, 12] and is otherwise called pseudo-Hermiticity [13–23].

An obvious generalization of the tachyonic case contains an imaginary mass and a  $\gamma^5$  mass term,

$$(i\gamma^\mu \partial_\mu - i m_1 - \gamma^5 m_2) \psi(x) = 0. \quad (1.9)$$

The dispersion relation is  $E = \sqrt{\vec{p}^2 - m_1^2 - m_2^2}$ . The corresponding Hamiltonian reads

$$H' = \vec{\alpha} \cdot \vec{p} + i\beta m_1 + \beta \gamma^5 m_2, \quad (1.10)$$

and is  $\gamma^5$  Hermitian,  $H' = \gamma^5 H'^+ \gamma^5$ . For  $m_2 = 0$ , Eq. (1.9) has been discussed in Refs. [24, 25].

It is our goal here to present the fundamental eigenspinors corresponding to the plane-wave solution of the equations (1.2), (1.4), (1.7), and (1.9) in a unified and systematic manner. Furthermore, we discuss the second-quantized versions of the fermionic theories described by the generalized Dirac equations. Anticipating part of the results, we may point out that the massless Dirac equation “interpolates” between the tardyonic equations (1.2) and (1.4) and the tachyonic equations (1.7) and (1.9). For zero mass, helicity and chirality are equal. Helicity and chirality “depart” from each other in very specific directions, when the tardyonic and tachyonic mass terms are “switched on”, as we shall discuss in the following.

## B. Tachyonic Dirac Equation and Neutrinos: Possible Connections

The tachyonic generalized Dirac equations (1.7) and (1.9) describe the motion of superluminal particles, which may either be important for astrophysical studies (neutrinos) or for artificially generated environments such as honeycomb photonic lattices in which pertinent dispersion relations become practically important [26]. Contrary to other, somewhat catchy, claims, the existence of superluminal particles would *not* falsify Einstein’s theory of special relativity [27], which according to common wisdom is based on the following postulates: (i) The principle of relativity states that the laws of physics are the same for all observers in uniform motion relative to one another. (ii) The speed of light in a vacuum is the same for all observers, regardless of their relative motion or of the motion of the source of the light. Predictions of relativity theory regarding the relativity of simultaneity, time dilation and length contraction would not change if superluminal particles did exist. Furthermore, as shown by Sudarshan *et al.* (Refs. [28–31]) and Feinberg (Refs. [32, 33]), the existence of tachyons, which are superluminal particles fulfilling a Lorentz-invariant dispersion relation  $E^2 = \vec{p}^2 - m_\nu^2$ , is fully compatible with special relativity and Lorentz invariance. According to special relativity, it is forbidden to accelerate a particle “through” the light barrier (because  $E = m/\sqrt{1-v^2} \rightarrow \infty$  for  $v \rightarrow 1$ ), but a genuinely superluminal particle remains superluminal upon Lorentz transformation. Significant problems are encountered when one attempts to quantize the tachyonic theories, but again, as shown in Ref. [9], these problems may not be as serious as previously thought. In particular, the so-called reinterpretation of solutions propagating into the past according to the Feynman prescription [31] is a cornerstone of modern field theory. Furthermore, it has been shown in Ref. [9] that tachyonic particles can be localized, and equal-time anticommutators of the spin-1/2 tachyonic field involve an unfiltered Dirac- $\delta$  [see Eq. (37) of Ref. [9]].

Despite these arguments, we can say that, from the point of view of fundamental symmetries, accepting a superluminal neutrino would be equivalent to an “ugly duckling”. Adding to the difficulties, we notice that recent experimental claims regarding the conceivable observation of highly superluminal neutrinos have turned out to be false. One may point out that a relative deviation  $v = c(1+\delta)$  with  $\delta \sim 10^{-5}$  at  $E \sim 30$  GeV, as claimed by some recent experimental collaborations, would correspond to a negative neutrino mass square in the order of  $\sim -(100 \text{ MeV})^2$ , if one assumes a Lorentz-invariant dispersion relation [34]. Still, there is at present no conclusive answer regarding the conceivable superluminality of at least one neutrino flavor [35–39], and it is intriguing that all available direct measurements of the neutrino mass square have resulted in negative expectation values, still compatible with zero within experimental uncertainty, whereas published experimental best estimates for the neutrino speed [40–43] have been superluminal, again still compatible with the speed of light within experimental error. The recent ICARUS result [43] is consistent with this trend [44–50]; the best estimate for the neutrino velocity is superluminal, but the deviation from  $v_\nu = c$  is statistically insignificant. The OPERA collaboration [51] has indicated a preliminary, revised result of  $(v - c)/c = (2.7 \pm 3.1(\text{stat})_{-2.8}^{+3.8}(\text{sys.})) \times 10^{-6}$ . Neither subluminal nor superluminal propagation velocities are excluded based on the available experimental data. The “ugly duckling of a superluminal neutrino” is not beautiful; if we are

to consider accepting it, then we should be able to hope that the emergence of at least one “swan” (or “intellectual benefit”) should be the result of this operation.

Before we discuss the possible emergence of these benefits, let us include some historic remarks. According to reliable sources [52], Professor J. A. Wheeler, in his later years at the University of Austin (Texas), used to argue that the neutrino has to be massless, necessarily, and that in his opinion, it could only be a massless Weyl particle with definite helicity (and chirality). Recently presented arguments [53] regarding the possibility of overtaking a subluminal, left-handed neutrino, looking back and seeing a right-handed neutrino, were supposedly already used by Wheeler in order to dispel the conceivable existence of a neutrino mass term. This paradox has been termed “autobahn paradox” in Ref. [53] and excludes a Dirac neutrino unless one assumes exotic processes like sterile right-handed massive neutrinos. [The problem with a right-handed sterile massive neutrino is that for massive neutrinos, chirality and helicity are different, hence a  $V - A$  coupling of the form  $\gamma^\mu(1 - \gamma^5)$  no longer vanishes for massive Dirac neutrinos if one uses the canonical eigenstates of the massive Dirac equation. One therefore has to invoke additional exotic mechanisms in order to ensure the “sterility” of the right-handed Dirac neutrinos.] Wheeler also disliked [52] the notion of a Majorana neutrino, arguing that the charge conjugation invariance condition imposed on the Majorana particle precludes the existence of plane-wave solutions to the Majorana equation, and maximally violates lepton number. Again, these arguments [54] are in full agreement with those recently given in Ref. [53].

The original standard model thus called for manifestly massless neutrinos. The commonly accepted observation of neutrino oscillations precludes the possibility that all three generations of neutrino mass eigenstates are massless. Lepton number conservation is based on the global gauge symmetry  $\psi \rightarrow \psi \exp(i\Lambda)$ , applied simultaneously to all lepton fields. A Majorana neutrino would destroy lepton number as a global symmetry but solve the “autobahn paradox”, because a Majorana neutrino would be equal to its own antiparticle and thus, looking back, the right-handed neutrino state would consist of the same particle=antiparticle.

On the other hand, if we assume that the neutrino is described by the tachyonic Dirac equation, then the following statements are valid:

- Statement #1: We can properly assign lepton number and use plane-wave eigenstates for incoming and outgoing particles, while allowing for nonvanishing mass terms and thus, mass square differences among the neutrino mass (not flavor) eigenstates.
- Statement #2: There is a natural resolution for the “autobahn paradox” because a left-handed spacelike neutrino always remains spacelike upon Lorentz transformation and cannot be overtaken.
- Statement #3: The right-handed particle and left-handed antiparticle states are suppressed due to negative Fock-space norm.
- Statement #4: At least qualitatively, tachyonic neutrinos could yield an explanation for a repulsive force on intergalactic distance scales as they are repulsed, like all tachyons, by gravitational interactions (“dark energy”).

Pauli [55] postulated the existence of neutrinos, on the basis of the conservation of angular momentum and energy, and also introduced pseudo-Hermitian operators [13]. Here, we describe conceivable connections of neutrino physics and pseudo-Hermitian operators. Final clarification can only come from experiment. When in 1956, F. Reines and C. Cowan [56] discovered the electron neutrino, two and a half years before Pauli’s death, Pauli replied [57] by telegram: “Thanks for message. Everything comes to him who knows how to wait. Pauli.” In defense of the tachyonic hypothesis, we would like to stress that a tachyonic Dirac neutrino would allow us to retain lepton number conservation as a symmetry of nature. We would thus like to write up these thoughts in the current article, with attention to detail. We should point out that our approach fully conserves Lorentz invariance, in contrast to the extensions of the Standard Model based on Lorentz-violating terms which can otherwise lead to superluminal propagation (see Tables 11 and 13 of Ref. [58]).

Units with  $\hbar = c = \epsilon_0 = 1$  are used throughout the paper. The organization is as follows: In Sec. II, we discuss massless and tardyonic theories. Tachyonic extensions of the Dirac equation are discussed in Sec. III. Connections of the fundamental tensor sums over the eigenspinors with the derivation of the time-ordered propagator are analyzed in Sec. IV. A candidate for dark energy is presented in Sec. V. Conclusions are reserved for Sec. VI.

## II. GENERALIZED DIRAC EQUATIONS: MASSLESS AND TARDYONIC THEORIES

### A. Massless Dirac Theory

The massless Dirac equation and the massless Dirac Hamiltonian read as

$$i\gamma^\mu \partial_\mu \psi(x) = 0, \quad H_0 = \vec{\alpha} \cdot \vec{p}. \quad (2.1)$$

We note that  $H_0$  is both Hermitian as well as  $\gamma^5$  Hermitian, i.e.,  $H_0 = \gamma^5 H_0^\dagger \gamma^5$ . The dispersion relation is  $E = |\vec{k}|$ . With  $k^\mu = (E, \vec{k})$ , we seek positive-energy and negative-energy solutions of the form

$$\psi(x) = u_\sigma(\vec{k}) \exp(-ik \cdot x), \quad \phi(x) = v_\sigma(\vec{k}) \exp(ik \cdot x), \quad (2.2)$$

where  $\sigma = \pm$  denotes a quantum number which is equal to the helicity for positive-energy states, and equal to the negative of the helicity for negative-energy states. With  $\not{k} = \gamma^\mu k_\mu$ , we have  $\not{k} u_\pm(k) = \not{k} v_\pm(k) = 0$ . In the massless limit, the solutions to the Dirac equation are given as (see Chap. 2 of Ref. [59])

$$u_+(\vec{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} a_+(\vec{k}) \\ a_+(\vec{k}) \end{pmatrix}, \quad u_-(\vec{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} a_-(\vec{k}) \\ -a_-(\vec{k}) \end{pmatrix}, \quad (2.3a)$$

$$v_+(\vec{k}) = -u_+(\vec{k}), \quad v_-(\vec{k}) = -u_-(\vec{k}). \quad (2.3b)$$

The well-known helicity spinors are recalled as

$$a_+(\vec{k}) = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) e^{i\varphi} \end{pmatrix}, \quad a_-(\vec{k}) = \begin{pmatrix} -\sin(\frac{\theta}{2}) e^{-i\varphi} \\ \cos(\frac{\theta}{2}) \end{pmatrix}. \quad (2.4)$$

These fulfill the fundamental relations  $(\vec{\sigma} \cdot \hat{\vec{k}}) a_\sigma(\vec{k}) = \sigma a_\sigma(\vec{k})$ , as well as  $\sum_\sigma a_\sigma(\vec{k}) \otimes a_\sigma^\dagger(\vec{k}) = \mathbb{1}_{2 \times 2}$ , and  $\sum_\sigma \sigma a_\sigma(\vec{k}) \otimes a_\sigma^\dagger(\vec{k}) = \vec{\sigma} \cdot \hat{\vec{k}}$ , where  $\hat{\vec{k}} = \vec{k}/|\vec{k}|$  and the sum over  $\sigma$  is over the values  $\pm 1$ . The sums over the fundamental bispinors  $u$  and  $v$  fulfill the following sum rules,

$$\text{Sum rule I:} \quad \boxed{\sum_\sigma 2 |\vec{k}| u_\sigma(\vec{k}) \otimes \bar{u}_\sigma(\vec{k}) = \not{k}, \quad \sum_\sigma 2 |\vec{k}| v_\sigma(\vec{k}) \otimes \bar{v}_\sigma(\vec{k}) = \not{k}}, \quad (2.5)$$

as well as

$$\text{Sum rule II:} \quad \boxed{\sum_\sigma 2 |\vec{k}| (-\sigma) u_\sigma(\vec{k}) \otimes \bar{u}_\sigma(\vec{k}) \gamma^5 = \not{k}, \quad \sum_\sigma 2 |\vec{k}| (-\sigma) v_\sigma(\vec{k}) \otimes \bar{v}_\sigma(\vec{k}) \gamma^5 = \not{k}}. \quad (2.6)$$

Sum rule I can be obtained by a quick explicit calculation, and sum rule II holds because in the massless limit, helicity equals  $\pm$ chirality (positive sign for positive energy, negative sign for negative energy). We denote the Dirac adjoint as  $\bar{u}_\sigma(\vec{k}) = u_\sigma^\dagger(\vec{k}) \gamma^0$ . One can easily check by an explicit calculation that  $\bar{u}_\sigma(\vec{k}) \gamma^5 = \left( \gamma^5 \gamma^0 u_\sigma(\vec{k}) \right)^\dagger = (-\sigma) \bar{u}_\sigma(\vec{k})$  and  $\bar{v}_\sigma(\vec{k}) \gamma^5 = \left( \gamma^5 \gamma^0 v_\sigma(\vec{k}) \right)^\dagger = (-\sigma) \bar{v}_\sigma(\vec{k})$ . We can thus introduce a factor  $(-\sigma)^2 = 1$  under the summation over spins in Eq. (2.5) and replace one of the factors  $(-\sigma)$  by a multiplication of the Dirac adjoint spinor from the right by the fifth current. The Lorentz-invariant normalization of the massless solutions vanishes, i.e.,  $\bar{u}_\sigma(\vec{k}) u_\sigma(\vec{k}) = \bar{v}_\sigma(\vec{k}) v_\sigma(\vec{k}) = 0$ .

Eigenstates of the massless Hamiltonian  $H_0 = \vec{\alpha} \cdot \vec{p}$  have to be eigenstates of the chirality operator  $\gamma^5$  because the chirality commutes with the Hamiltonian, in the sense that  $[\gamma^5, H_0] = 0$ . Furthermore,

$$H_0 = \vec{\alpha} \cdot \vec{p} = |\vec{p}| \gamma^5 \left( \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} \right), \quad (2.7)$$

where  $\vec{\Sigma} = \gamma^5 \vec{\alpha}$  is the vector of  $4 \times 4$  spin matrices, and the helicity operator is identified as  $\vec{\Sigma} \cdot \vec{p}/|\vec{p}|$ . Let  $\lambda_1$  be the eigenvalue of chirality and  $\lambda_2$  be the eigenvalue of the helicity operator. Then, the eigenvalue of the Hamiltonian is  $E_0 = |\vec{p}| \lambda_1 \lambda_2$ . Since  $\lambda_1 = \pm 1$  and  $\lambda_2 = \pm 1$ , we easily recover the known fact that helicity equals chirality for positive energy, whereas the relation is reversed for negative-energy states (see also Chap. 2.4 of Ref. [59]). We are aware of the fact that the considerations reported in the current section partly refer to the literature but we give them in some detail because they are essential for the following considerations.

## B. Massive Dirac Theory

We start from the ordinary Dirac equation given in Eq. (1.2), which reads  $(i\gamma^\mu \partial_\mu - m_1) \psi(x) = 0$ . In the helicity basis, the fundamental spinor solutions read as

$$\psi(x) = U_\pm^{(1)}(\vec{k}) \exp(-ik \cdot x), \quad \phi(x) = V_\pm^{(1)}(\vec{k}) \exp(ik \cdot x), \quad (2.8)$$

The algebraic relations that have to be fulfilled by the bispinor amplitudes  $U_{\pm}^{(1)}(\vec{k})$  and  $V_{\pm}^{(1)}(\vec{k})$  read as follows,

$$(\not{k} - m_1) U_{\pm}^{(1)}(\vec{k}) = 0, \quad (\not{k} + m_1) V_{\pm}^{(1)}(\vec{k}) = 0. \quad (2.9)$$

The dispersion relation is  $E^{(1)} = \sqrt{\vec{k}^2 + m_1^2}$ . In the helicity basis, the solutions of the equation (2.9) with a tardyonic  $m_1$  mass term are easily written down, using the identity  $(\not{k} - m_1)(\not{k} + m_1) = k^2 - m_1^2 = 0$ . With an appropriate normalization factor, and after some algebraic simplification, the positive-energy solutions read as follows,

$$U_+^{(1)}(\vec{k}) = \frac{(\not{k} + m_1) u_+(\vec{k})}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}} = \begin{pmatrix} \sqrt{\frac{E^{(1)} + m_1}{2 E^{(1)}}} a_+(\vec{k}) \\ \sqrt{\frac{E^{(1)} - m_1}{2 E^{(1)}}} a_+(\vec{k}) \end{pmatrix}, \quad (2.10a)$$

$$U_-^{(1)}(\vec{k}) = \frac{(\not{k} + m_1) u_-(\vec{k})}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}} = \begin{pmatrix} \sqrt{\frac{E^{(1)} + m_1}{2 E^{(1)}}} a_-(\vec{k}) \\ -\sqrt{\frac{E^{(1)} - m_1}{2 E^{(1)}}} a_-(\vec{k}) \end{pmatrix}. \quad (2.10b)$$

The negative-energy eigenstates of the tardyonic equations in the helicity basis are given as

$$V_+^{(1)}(\vec{k}) = \frac{(m_1 - \not{k}) v_+(\vec{k})}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}} = \begin{pmatrix} -\sqrt{\frac{E^{(1)} - m_1}{2 E^{(1)}}} a_+(\vec{k}) \\ -\sqrt{\frac{E^{(1)} + m_1}{2 E^{(1)}}} a_+(\vec{k}) \end{pmatrix}, \quad (2.11a)$$

$$V_-^{(1)}(\vec{k}) = \frac{(m_1 - \not{k}) v_-(\vec{k})}{\sqrt{(E^{(1)} - |\vec{k}|)^2 + m_1^2}} = \begin{pmatrix} -\sqrt{\frac{E^{(1)} - m_1}{2 E^{(1)}}} a_-(\vec{k}) \\ \sqrt{\frac{E^{(1)} + m_1}{2 E^{(1)}}} a_-(\vec{k}) \end{pmatrix}. \quad (2.11b)$$

These solutions are consistent with those given in Ref. [3] and in Chap. 23 of Ref. [60], and the normalizations are ( $\sigma = \pm$ )

$$U_{\sigma}^{(1)+}(\vec{k}) U_{\sigma}^{(1)}(\vec{k}) = V_{\sigma}^{(1)+}(\vec{k}) V_{\sigma}^{(1)}(\vec{k}) = 1. \quad (2.12)$$

One can change the normalization according to

$$\mathcal{U}_{\sigma}^{(1)}(\vec{k}) = \left(\frac{E^{(1)}}{m_1}\right)^{1/2} U_{\sigma}^{(1)}(\vec{k}), \quad \mathcal{V}_{\sigma}^{(1)}(\vec{k}) = \left(\frac{E^{(1)}}{m_1}\right)^{1/2} V_{\sigma}^{(1)}(\vec{k}). \quad (2.13)$$

The Lorentz-invariant normalization is equal to one for the fundamental positive-energy bispinors and equal to minus one for the fundamental negative-energy bispinors,

$$\bar{\mathcal{U}}_{\sigma}^{(1)}(\vec{k}) \mathcal{U}_{\sigma}^{(1)}(\vec{k}) = 1, \quad \bar{\mathcal{V}}_{\sigma}^{(1)}(\vec{k}) \mathcal{V}_{\sigma}^{(1)}(\vec{k}) = -1. \quad (2.14)$$

A little algebra is sufficient to reproduce the following known sums over bispinors,

$$\boxed{\sum_{\sigma} \mathcal{U}_{\sigma}^{(1)}(\vec{k}) \otimes \bar{\mathcal{U}}_{\sigma}^{(1)}(\vec{k}) = \frac{\not{k} + m_1}{2m_1}}, \quad \boxed{\sum_{\sigma} \mathcal{V}_{\sigma}^{(1)}(\vec{k}) \otimes \bar{\mathcal{V}}_{\sigma}^{(1)}(\vec{k}) = \frac{\not{k} - m_1}{2m_1}}. \quad (2.15a)$$

In accordance with general wisdom about the tardyonic case, these do not involve a helicity-dependent prefactor. The sum rule (2.15) is of type I [see Eq. (2.5)].

### C. Two Tardyonic Mass Terms

Inspired by the discussion in Sec. I, we consider an equation with two tardyonic mass terms, which has already been indicated in Eq. (1.2) and reads  $(i\gamma^\mu \partial_\mu - m_1 - i\gamma^5 m_2) \psi^{(t)}(x) = 0$ . For the corresponding bispinors in the fundamental plane-wave solutions, this implies that

$$(\not{k} - m_1 - i\gamma^5 m_2) U_\pm^{(t)}(\vec{k}) = 0, \quad (2.16a)$$

$$(-\not{k} - m_1 - i\gamma^5 m_2) V_\pm^{(t)}(\vec{k}) = 0. \quad (2.16b)$$

The dispersion relation is  $E^{(t)} = \sqrt{\vec{k}^2 + m_1^2 + m_2^2}$ . The fundamental positive-energy bispinors read as follows,

$$U_+^{(t)}(\vec{k}) = \frac{(\not{k} + m_1 - i\gamma^5 m_2) u_+(\vec{k})}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} = \begin{pmatrix} \frac{m_1 - i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \\ \frac{m_1 - i m_2 - E^{(t)} + |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \end{pmatrix}, \quad (2.17a)$$

$$U_-^{(t)}(\vec{k}) = \frac{(\not{k} + m_1 - i\gamma^5 m_2) u_-(\vec{k})}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} = \begin{pmatrix} \frac{m_1 + i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \\ \frac{-m_1 - i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \end{pmatrix}. \quad (2.17b)$$

The negative-energy eigenstates of the equation with two tardyonic mass terms are given as

$$V_+^{(t)}(\vec{k}) = \frac{(-\not{k} - i\gamma^5 m_2 + m_1) v_+(\vec{k})}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} = \begin{pmatrix} \frac{-m_1 + i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \\ \frac{-m_1 + i m_2 - E^{(t)} + |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \end{pmatrix},$$

for negative helicity (positive chirality in the massless limit) and

$$V_-^{(t)}(\vec{k}) = \frac{(-\not{k} - i\gamma^5 m_2 + m_1) v_-(\vec{k})}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} = \begin{pmatrix} \frac{-m_1 - i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \\ \frac{m_1 + i m_2 + E^{(t)} - |\vec{k}|}{\sqrt{(E^{(t)} - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \end{pmatrix}.$$

for positive helicity (negative chirality in the massless limit). In the massless limit (first  $E^{(t)} \rightarrow |\vec{k}|$ , then  $m_2 \rightarrow 0$ , and then  $m_1 \rightarrow 0$ ), we again reproduce the massless solutions,  $U_+^{(t)}(\vec{k}) \rightarrow u_+(\vec{k})$ ,  $U_-^{(t)}(\vec{k}) \rightarrow u_-(\vec{k})$ ,  $V_+^{(t)}(\vec{k}) \rightarrow v_+(\vec{k})$  and  $V_-^{(t)}(\vec{k}) \rightarrow v_-(\vec{k})$ . Of course, in the limit  $m_1 \rightarrow 0$ , one also has to expand the normalization denominators in powers of  $m_1$ . For  $m_2 = 0$  the solutions (2.17) and (2.18) reduce to the solutions of the ordinary Dirac equation in Eqs. (2.10) and (2.11), and can be expanded in  $m_2$  to yield corrections to the ordinary Dirac equation for small  $m_2$ , i.e  $m_1 \gg m_2$ . The states are normalized with respect to the condition

$$U_\sigma^{(t)}(\vec{k}) U_\sigma^{(t)}(\vec{k}) = V_\sigma^{(t)}(\vec{k}) V_\sigma^{(t)}(\vec{k}) = 1. \quad (2.19)$$

In the normalization

$$\mathcal{U}_\sigma^{(t)}(\vec{k}) = \left(\frac{E^{(t)}}{m_1}\right)^{1/2} U_\sigma^{(t)}(\vec{k}), \quad \mathcal{V}_\sigma^{(t)}(\vec{k}) = \left(\frac{E^{(t)}}{m_1}\right)^{1/2} V_\sigma^{(t)}(\vec{k}), \quad (2.20a)$$

the positive-energy solutions acquire a “positive Lorentz-invariant norm”, whereas the negative-energy solutions have “negative Lorentz-invariant norm”,

$$\overline{\mathcal{U}}_\sigma^{(t)}(\vec{k}) \mathcal{U}_\sigma^{(t)}(\vec{k}) = 1, \quad \overline{\mathcal{V}}_\sigma^{(t)}(\vec{k}) \mathcal{V}_\sigma^{(t)}(\vec{k}) = -1. \quad (2.21)$$



After some algebra, one can derive the following sums over bispinors,

$$\boxed{\sum_{\sigma} \mathcal{U}_{\sigma}^{(t)}(\vec{k}) \otimes \overline{\mathcal{U}}_{\sigma}^{(t)}(\vec{k}) = \frac{\not{k} + m_1 - i\gamma^5 m_2}{2m_1}}, \quad \boxed{\sum_{\sigma} \mathcal{V}_{\sigma}^{(t)}(\vec{k}) \otimes \overline{\mathcal{V}}_{\sigma}^{(t)}(\vec{k}) = \frac{\not{k} - m_1 + i\gamma^5 m_2}{2m_1}}. \quad (2.22)$$

These are easily identified as the positive- and negative-energy projectors. The sum rules do not involve helicity-dependent prefactor and are of type I [see Eq. (2.5)].

The solutions (2.17) and (2.18) approach the massless solutions if one replaces  $m_2 \rightarrow 0$  first and then lets  $m_1 \rightarrow 0$ . They are thus useful for systems where the  $m_1$  mass is greater than  $m_2$ . For  $m_2 \gg m_1$ , one would like to calculate solutions that approach the massless case for the sequence  $m_1 \rightarrow 0$ , then  $m_2 \rightarrow 0$ . These read as follows,

$$U_{\sigma}^{\prime(t)}(\vec{k}) = i\sigma U_{\sigma}^{(t)}(\vec{k}), \quad V_{\sigma}^{\prime(t)}(\vec{k}) = i\sigma V_{\sigma}^{(t)}(\vec{k}). \quad (2.23)$$

In comparison to the solutions (2.17) and (2.18), they acquire a nontrivial phase factor.

### III. GENERALIZED DIRAC EQUATIONS: TACHYONIC MASS TERMS

#### A. Tachyonic Dirac Equation

The tachyonic Dirac equation is given in Eq. (1.7) and reads  $(i\gamma^{\mu} \partial_{\mu} - \gamma^5 m) \psi(x) = 0$ . The fundamental bispinors entering the equations fulfill the equations

$$(\not{k} - \gamma^5 m) U_{\pm}(\vec{k}) = 0, \quad (\not{k} + \gamma^5 m) V_{\pm}(\vec{k}) = 0. \quad (3.1)$$

Using  $(\not{k} - \gamma^5 m)(\not{k} - \gamma^5 m) = k^2 + m^2$  and some algebra, the prefactors in the fundamental bispinors (for positive energy) take a very simple form,

$$U_{+}(\vec{k}) = \frac{(\gamma^5 m - \not{k}) u_{+}(\vec{k})}{\sqrt{(E - |\vec{k}|)^2 + m^2}} = \begin{pmatrix} \sqrt{\frac{|\vec{k}| + m}{2|\vec{k}|}} a_{+}(\vec{k}) \\ \sqrt{\frac{|\vec{k}| - m}{2|\vec{k}|}} a_{+}(\vec{k}) \end{pmatrix}, \quad (3.2a)$$

$$U_{-}(\vec{k}) = \frac{(\not{k} - \gamma^5 m) u_{-}(\vec{k})}{\sqrt{(E - |\vec{k}|)^2 + m^2}} = \begin{pmatrix} \sqrt{\frac{|\vec{k}| - m}{2|\vec{k}|}} a_{-}(\vec{k}) \\ -\sqrt{\frac{|\vec{k}| + m}{2|\vec{k}|}} a_{-}(\vec{k}) \end{pmatrix}. \quad (3.2b)$$

For negative energy, the solutions read as follows,

$$V_{+}(\vec{k}) = \frac{(\gamma^5 m + \not{k}) v_{+}(\vec{k})}{\sqrt{(E - |\vec{k}|)^2 + m^2}} = \begin{pmatrix} -\sqrt{\frac{|\vec{k}| - m}{2|\vec{k}|}} a_{+}(\vec{k}) \\ -\sqrt{\frac{|\vec{k}| + m}{2|\vec{k}|}} a_{+}(\vec{k}) \end{pmatrix}, \quad (3.3a)$$

$$V_{-}(\vec{k}) = \frac{(-\not{k} - \gamma^5 m) v_{-}(\vec{k})}{\sqrt{(E - |\vec{k}|)^2 + m^2}} = \begin{pmatrix} -\sqrt{\frac{|\vec{k}| + m}{2|\vec{k}|}} a_{-}(\vec{k}) \\ \sqrt{\frac{|\vec{k}| - m}{2|\vec{k}|}} a_{-}(\vec{k}) \end{pmatrix}. \quad (3.3b)$$

The normalization condition is  $(\sigma = \pm)$

$$U_{\sigma}^{+}(\vec{k}) U_{\sigma}(\vec{k}) = V_{\sigma}^{+}(\vec{k}) V_{\sigma}(\vec{k}) = 1. \quad (3.4)$$



One can change to Lorentz-invariant normalization by a multiplication with  $(|\vec{k}|/m)^{1/2}$ ,

$$\mathcal{U}_\sigma(\vec{k}) = \left(\frac{|\vec{k}|}{m}\right)^{1/2} U_\sigma(\vec{k}), \quad \mathcal{V}_\sigma(\vec{k}) = \left(\frac{|\vec{k}|}{m}\right)^{1/2} V_\sigma(\vec{k}). \quad (3.5)$$

The “calligraphic” spinors fulfill the following helicity-dependent normalizations,

$$\overline{\mathcal{U}}_\sigma(\vec{k}) \mathcal{U}_\sigma(\vec{k}) = \sigma, \quad \overline{\mathcal{V}}_\sigma(\vec{k}) \mathcal{V}_\sigma(\vec{k}) = -\sigma, \quad (3.6)$$

where we observe that  $\sigma$  is a good quantum number because the helicity operator commutes with the Hamiltonian (1.8). The sum rule fulfilled by the fundamental plane-wave spinors is of type II [see Eq. (2.6)]. For the positive-energy spinors, we have

$$\sum_{\sigma} (-\sigma) \mathcal{U}_\sigma(\vec{k}) \otimes \overline{\mathcal{U}}_\sigma(\vec{k}) \gamma^5 = \frac{\not{k} - \gamma^5 m}{2m}, \quad (3.7a)$$

where for the negative-energy spinors, the sum rule reads

$$\sum_{\sigma} (-\sigma) \mathcal{V}_\sigma(\vec{k}) \otimes \overline{\mathcal{V}}_\sigma(\vec{k}) \gamma^5 = \frac{\not{k} + \gamma^5 m}{2m}. \quad (3.7b)$$

The expressions on the right-hand sides are the positive- and negative-energy projectors.

## B. Two Tachyonic Mass Terms

We study the equation  $(i\gamma^\mu \partial_\mu - im_1 - \gamma^5 m_2) \psi(x) = 0$ , as given in Eq. (1.9). The fundamental spinors, which we denote as  $U'_\pm(\vec{k})$  and  $V'_\pm(\vec{k})$ , fulfill the following equations,

$$(\not{k} - im_1 - \gamma^5 m_2) U'_\pm(\vec{k}) = 0, \quad (3.8a)$$

$$(\not{k} + im_1 + \gamma^5 m_2) V'_\pm(\vec{k}) = 0. \quad (3.8b)$$

The positive-energy solutions are obtained using the identity  $(\not{k} - im_1 - \gamma^5 m_2)(\not{k} + im_1 - \gamma^5 m_2) = k^2 + m_1^2 + m_2^2$ . With  $E' = \sqrt{\vec{k}^2 - m_1^2 - m_2^2}$ , they read as follows,

$$U'_+(\vec{k}) = \begin{pmatrix} \frac{im_1 + m_2 - E' + |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \\ \frac{im_1 + m_2 + E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \end{pmatrix}, \quad (3.9a)$$

$$U'_-(\vec{k}) = \begin{pmatrix} \frac{im_1 + m_2 + E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \\ \frac{-im_1 - m_2 + E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \end{pmatrix}. \quad (3.9b)$$

The negative-energy solutions for the tachyonic equation with two mass terms, are given as

$$V'_+(\vec{k}) = \begin{pmatrix} \frac{\mathrm{i}m_1 - m_2 - E' + |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \\ \frac{\mathrm{i}m_1 - m_2 - E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_+(\vec{k})}{\sqrt{2}} \end{pmatrix}, \quad (3.10a)$$

$$V'_-(\vec{k}) = \begin{pmatrix} \frac{-\mathrm{i}m_1 - m_2 + E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \\ \frac{\mathrm{i}m_1 + m_2 + E' - |\vec{k}|}{\sqrt{(E' - |\vec{k}|)^2 + m_1^2 + m_2^2}} \frac{a_-(\vec{k})}{\sqrt{2}} \end{pmatrix}. \quad (3.10b)$$

The normalization condition is  $U'^{+}_{\sigma}(\vec{k}) U'_{\sigma}(\vec{k}) = V'^{+}_{\sigma}(\vec{k}) V'_{\sigma}(\vec{k}) = 1$ . We use a definition of the ‘‘calligraphic’’ spinors analogous to Eq. (3.5),

$$\mathcal{U}'_{\sigma}(\vec{k}) = \left( \frac{|\vec{k}|}{m_2} \right)^{1/2} U'_{\sigma}(\vec{k}), \quad \mathcal{V}'_{\sigma}(\vec{k}) = \left( \frac{|\vec{k}|}{m_2} \right)^{1/2} V'_{\sigma}(\vec{k}). \quad (3.11)$$

In analogy to Eq. (3.7), a sum rule of type II [see Eq. (2.6)] is fulfilled by the fundamental plane-wave spinors,

$$\boxed{\sum_{\sigma} (-\sigma) \mathcal{U}'_{\sigma}(\vec{k}) \otimes \overline{\mathcal{U}}'_{\sigma}(\vec{k}) \gamma^5 = \frac{\not{k} + \mathrm{i}m_1 - \gamma^5 m_2}{2m_2}}, \quad (3.12)$$

$$\boxed{\sum_{\sigma} (-\sigma) \mathcal{V}'_{\sigma}(\vec{k}) \otimes \overline{\mathcal{V}}'_{\sigma}(\vec{k}) \gamma^5 = \frac{\not{k} - \mathrm{i}m_1 + \gamma^5 m_2}{2m_2}}.$$

We thus obtain the desired projectors onto positive- and negative-energy solutions for the Dirac equation with two tachyonic mass terms (1.9). The generalized equation  $(\mathrm{i}\gamma^{\mu}\partial_{\mu} - \mathrm{i}m_1 - \gamma^5 m_2)\psi(x) = 0$  is fully compatible with the Clifford-algebra based approach recently described in Ref. [61].

## IV. THEOREMS FOR GENERALIZED DIRAC FIELDS

### A. Spinor Sums and Time-Ordered Propagator

Our central postulate regarding the quantized fermionic theory is that the time-ordered vacuum expectation value of the field operators should yield the time-ordered (Feynman) propagator, which, in the momentum representation, is equal to the inverse of the Hamiltonian (upon multiplication with  $\gamma^0$ ). This postulate implies that under rather general assumptions regarding the mathematical form of the elementary field anticommutators, sum rules have to be fulfilled by the tensor sums over the fundamental spinor solutions. It is perhaps not surprising that these sum rules, are precisely of the form investigated in Secs. II and III of this paper.

For definiteness, we consider the solution of the tachyonic Dirac equation (Sec. III A). The generalization to other generalized Dirac equations is straightforward. We start from the field operator [9]

$$\psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{m}{E} \sum_{\sigma=\pm} \left\{ b_{\sigma}(k) \mathcal{U}_{\sigma}(\vec{k}) e^{-\mathrm{i}k \cdot x} + d_{\sigma}^{\dagger}(k) \mathcal{V}_{\sigma}(\vec{k}) e^{\mathrm{i}k \cdot x} \right\},$$

$$k = (E, \vec{k}), \quad E = E_{\vec{k}} = \sqrt{\vec{k}^2 - m^2 - \mathrm{i}\epsilon}, \quad (4.1)$$

where  $b_{\sigma}$  annihilates particles and  $d_{\sigma}^{\dagger}$  creates antiparticles. The following anticommutators vanish,

$$\{b_{\sigma}(k), b_{\rho}(k')\} = \{b_{\sigma}^{\dagger}(k), b_{\rho}^{\dagger}(k')\} = 0, \quad (4.2a)$$

$$\{d_{\sigma}(k), d_{\rho}(k')\} = \{d_{\sigma}^{\dagger}(k), d_{\rho}^{\dagger}(k')\} = 0, \quad (4.2b)$$

We assume the following general form for the nonvanishing anticommutators,

$$\{b_\sigma(k), b_\rho^+(k')\} = f(\sigma, \vec{k}) (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}, \quad (4.3a)$$

$$\{d_\sigma(k), d_\rho^+(k')\} = g(\sigma, \vec{k}) (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}, \quad (4.3b)$$

with arbitrary  $f$  and  $g$  functions of the quantum numbers  $\sigma$  and  $\vec{k}$ . One might argue that since  $f$  and  $g$  must be dimensionless, they can depend only on the dimensionless arguments  $\sigma$  and  $\vec{k}/m$ , but that is a detail of the discussion which we do not pursue any further. Our only assumption concerns the fact that the field anticommutators should be diagonal in the helicity and wave vector quantum numbers, leading to the corresponding Kronecker and Dirac- $\delta$ 's.

We assume that the spin-matrix  $\Gamma$  either constitutes a Lorentz scalar or a pseudo-scalar quantity, which is a scalar under the proper orthochronous Lorentz group. The time-ordered product of field operators reads as

$$\begin{aligned} \langle 0 | T \psi(x) \bar{\psi}(y) \Gamma | 0 \rangle = & \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \left\{ \Theta(x^0 - y^0) e^{-ik \cdot (x-y)} \sum_{\sigma=\pm} f(\sigma, \vec{k}) \mathcal{U}_\sigma(\vec{k}) \otimes \bar{\mathcal{U}}_\sigma(\vec{k}) \Gamma \right. \\ & \left. - \Theta(y^0 - x^0) e^{ik \cdot (x-y)} \sum_{\sigma=\pm} g(\sigma, \vec{k}) \mathcal{V}_\sigma(\vec{k}) \otimes \bar{\mathcal{V}}_\sigma(\vec{k}) \Gamma \right\}. \end{aligned} \quad (4.4)$$

This equation contains the same coefficient functions  $f$  and  $g$  that enter into Eq. (4.3). In order to proceed with the derivation of the propagator, we must postulate that the following sum rules hold,

$$\sum_{\sigma} f(\sigma, \vec{k}) \mathcal{U}_\sigma(\vec{k}) \otimes \bar{\mathcal{U}}_\sigma(\vec{k}) \Gamma = \frac{\not{k} - \gamma^5 m}{2m}, \quad \sum_{\sigma} g(\sigma, \vec{k}) \mathcal{V}_\sigma(\vec{k}) \otimes \bar{\mathcal{V}}_\sigma(\vec{k}) \Gamma = \frac{\not{k} + \gamma^5 m}{2m}. \quad (4.5)$$

The sum rule (4.5) is crucial for the further steps in the derivation of the time-ordered propagator. Introducing a suitable complex integral representation for the step function, one obtains from Eq. (4.4), using Eq. (4.5), after a few steps which we do not discuss in further detail [see also Eqs. (3.169) and Eq. (3.170) of Ref. [59]]

$$\begin{aligned} \langle 0 | T \psi(x) \bar{\psi}(y) \Gamma | 0 \rangle = & i \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \int \frac{dk_0}{2\pi} e^{-ik_0 \cdot (x^0 - y^0) + i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{\gamma^0 k_0 - \vec{\gamma} \cdot \vec{k} - \gamma^5 m}{2m(k_0 - E + i\epsilon)} \\ & + i \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \int \frac{dk_0}{2\pi} e^{-ik_0 \cdot (x^0 - y^0) + i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{-\gamma^0 k_0 + \vec{\gamma} \cdot \vec{k} + \gamma^5 m}{2m(k_0 + E - i\epsilon)}. \end{aligned} \quad (4.6)$$

The convention is that in any integrals  $\int d^3k$ , the component  $k_0$  is set equal to  $E = \sqrt{\vec{k}^2 + m^2}$  in the integrand when it occurs in scalar products of the form  $k \cdot (x - y)$  etc., but if the integral is over the full  $d^4k$ , then the integration interval is the full  $k_0 \in (-\infty, \infty)$ . With the convention which is adopted in many quantum field theoretical textbooks, including Refs. [59, 62], we finally obtain the result

$$\langle 0 | T \psi(x) \bar{\psi}(y) \Gamma | 0 \rangle = i \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{\not{k} - \gamma^5 m}{k^2 + m^2 + i\epsilon}. \quad (4.7)$$

The tachyonic propagator  $S_T$  is identified, under the integral sign, as

$$S_T(k) = \frac{1}{\not{k} - \gamma^5 (m + i\epsilon)} = \frac{\not{k} - \gamma^5 m}{k^2 + m^2 + i\epsilon}. \quad (4.8)$$

The sum rule (3.7) implies that the derivation is valid for the choice

$$f(\sigma, \vec{k}) = g(\sigma, \vec{k}) = -\sigma, \quad \Gamma = \gamma^5, \quad (4.9)$$

in which case the relations given in Eq. (4.5) are fulfilled. Note that this observation does not imply that the choice (4.9) necessarily is the only one for which we are able to fulfill the postulates given in Eq. (4.5), but it is the only structurally simple choice that we have found.

For the egregiously simple choice (4.9), let us study the transition to the massless limit (2.6) in some further detail. Indeed, in the limit  $m \rightarrow 0$ , the denominator of the spin sums in Eq. (3.7) vanishes, and a finite limit is obtained after

multiplication with  $2m$ ,

$$\lim_{m \rightarrow 0} \sum_{\sigma} 2m (-\sigma) \mathcal{U}_{\sigma}(\vec{k}) \otimes \overline{\mathcal{U}}_{\sigma}(\vec{k}) \gamma^5 = \not{k}, \quad (4.10a)$$

$$\lim_{m \rightarrow 0} \sum_{\sigma} 2m (-\sigma) \mathcal{V}_{\sigma}(\vec{k}) \otimes \overline{\mathcal{V}}_{\sigma}(\vec{k}) \gamma^5 = \not{k}. \quad (4.10b)$$

In order to compare the normalizations of the fundamental spinors in the massless limit, we calculate the following quantities,

$$\overline{\mathcal{U}}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 \mathcal{U}_{\sigma}(\vec{k}) = -\sigma \frac{E}{m}, \quad \overline{u}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 u_{\sigma}(\vec{k}) = -\sigma, \quad (4.11a)$$

$$\overline{\mathcal{V}}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 \mathcal{V}_{\sigma}(\vec{k}) = -\sigma \frac{E}{m}, \quad \overline{v}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 v_{\sigma}(\vec{k}) = -\sigma, \quad (4.11b)$$

where the fundamental spinors  $u_{\sigma}$  and  $v_{\sigma}$  of the massless equation have been given in Sec. II A. Observing that  $E = |\vec{k}|$  in the massless limit, the identifications  $\sqrt{m} \mathcal{U}_{\sigma}(\vec{k}) \rightarrow \sqrt{|\vec{k}|} u_{\sigma}(\vec{k})$  and  $\sqrt{m} \mathcal{V}_{\sigma}(\vec{k}) \rightarrow \sqrt{|\vec{k}|} v_{\sigma}(\vec{k})$ , as implied by Eq. (4.11), show that the identities (4.10) precisely reduce to the sum rule (2.6) in the massless limit.

## B. Generalized Field Anticommutators for Tardyonic and Tachyonic Fields

For definiteness, we have considered the case of the tachyonic Dirac field in the above derivation. The decisive observation is that the choice

$$\text{tachyonic choice: } f(\sigma, \vec{k}) = g(\sigma, \vec{k}) = -\sigma, \quad \Gamma = \gamma^5, \quad (4.12)$$

is consistent with both massive tachyonic fields discussed in Secs. III A and III B, whereas the tardyonic choice

$$\text{tardyonic choice: } f(\sigma, \vec{k}) = g(\sigma, \vec{k}) = 1, \quad \Gamma = \mathbb{1}_{4 \times 4}, \quad (4.13)$$

yields the time-ordered propagator for both massive tardyonic fields discussed in Sec. II B and II C. The nonvanishing anticommutators for tardyons take the simple form [cf. Eq. (4.3)],

tardyonic anticommutators:

$$\{b_{\sigma}(k), b_{\rho}^{+}(k')\} = (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}, \quad \{d_{\sigma}(k), d_{\rho}^{+}(k')\} = (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}. \quad (4.14)$$

Again, compared with Eq. (4.3), the nonvanishing anticommutators for tachyons take the simple form

tachyonic anticommutators:

$$\{b_{\sigma}(k), b_{\rho}^{+}(k')\} = (-\sigma) (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}, \quad \{d_{\sigma}(k), d_{\rho}^{+}(k')\} = (-\sigma) (2\pi)^3 \frac{E}{m} \delta^3(\vec{k} - \vec{k}') \delta_{\sigma\rho}. \quad (4.15)$$

With these universal choices, the theory of the tardyonic and tachyonic spin-1/2 fields can be unified. The time-ordered propagator is given as

$$\langle 0 | T \psi(x) \overline{\psi}(y) \Gamma | 0 \rangle = i S(x - y), \quad S(x - y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x - y)} S(k). \quad (4.16)$$

We use the sum rules for the tensor sums over fundamental spinors given in Eq. (2.15) (for the tardyonic Dirac field), Eq. (2.22) (for the tardyonic Dirac field with two mass terms), Eq. (3.7) (for the tachyonic Dirac field) and in Eq. (3.12) (for the Dirac field of imaginary mass). Going through the exact same derivation as outlined above in between Eqs. (4.4) and (4.8), we obtain the following results for the time-ordered propagators of tardyonic and tachyonic fields. For the tardyonic Dirac field (Sec. II B), one has

$$S^{(1)}(k) = \frac{1}{\not{k} - m_1 + i\epsilon} = \frac{\not{k} + m_1}{k^2 - m_1^2 + i\epsilon}. \quad (4.17)$$

For the tardyonic field with two mass terms (Sec. (II C)), the Feynman propagator is easily found as

$$S^{(t)}(k) = \frac{1}{\not{k} - m_1 + i\epsilon - i\gamma^5(m_2 - i\eta)} = \frac{\not{k} + m_1 - i\gamma^5 m_2}{k^2 - m_1^2 - m_2^2 + i\epsilon}, \quad (4.18)$$

where  $\epsilon$  and  $\eta$  are infinitesimal imaginary parts. Both tardyonic mass terms acquire an infinitesimal negative imaginary part, and the prefactor  $m/E$  from Eq. (4.1) in the field operator needs to be replaced by  $m_1/E^{(t)}$  where the tardyonic energy is  $E^{(t)} = \sqrt{\vec{k}^2 + m_1^2 + m_2^2}$ . For the tachyonic Dirac field (Sec. III A and Ref. [9]), one has the result given in Eq. (4.8). Finally, for the Dirac field with two tachyonic mass terms, we have

$$S'(k) = \frac{\not{k} + i m_1 - \gamma^5 m_2}{k^2 + m_1^2 + m_2^2 + i\epsilon}. \quad (4.19)$$

In the latter case, the prefactor  $m/E$  from Eq. (4.1) in the field operator needs to be replaced by  $m_1/E'$  where the tachyonic energy is  $E' = \sqrt{\vec{k}^2 - m_1^2 - m_2^2}$ . For both tachyonic fields discussed here, the mass acquires an infinitesimal positive imaginary part, as manifest in the results given in Eqs. (4.8) and (4.19).

### C. Tachyonic Gordon Identities

It is useful to illustrate the derivation outlined above by exploring its connection to tachyonic Gordon identities. For definiteness, we again concentrate on the tachyonic Dirac equation discussed in Sec. III A. The matrix element of the vector current finds the following Gordon decomposition for positive-energy spinors,

$$\bar{U}_\pm(\vec{k}') \gamma^\mu U_\pm(\vec{k}) = \frac{1}{2m} \bar{U}_\pm(\vec{k}') \gamma^5 [(k'^\mu - k^\mu) + i\sigma^{\mu\nu}(k'_\nu + k_\nu)] U_\pm(\vec{k}).$$

For negative-energy solutions, the identity reads as

$$\bar{V}_\pm(\vec{k}') \gamma^\mu V_\pm(\vec{k}) = -\frac{1}{2m} \bar{V}_\pm(\vec{k}') \gamma^5 [(k'^\mu - k^\mu) + i\sigma^{\mu\nu}(k'_\nu + k_\nu)] V_\pm(\vec{k}).$$

For  $k' = k$ , one has

$$\bar{U}_\pm(\vec{k}) \gamma^\mu U_\pm(\vec{k}) = \frac{i}{m} \bar{U}_\pm(\vec{k}) \gamma^5 \sigma^{\mu\nu} k_\nu U_\pm(\vec{k}), \quad (4.21a)$$

$$\bar{V}_\pm(\vec{k}) \gamma^\mu V_\pm(\vec{k}) = -\frac{i}{m} \bar{V}_\pm(\vec{k}) \gamma^5 \sigma^{\mu\nu} k_\nu V_\pm(\vec{k}). \quad (4.21b)$$

The matrix element of the axial current reads

$$\bar{U}_\pm(\vec{k}') \gamma^5 \gamma^\mu U_\pm(\vec{k}) = -\frac{1}{2m} \bar{U}_\pm(\vec{k}') [(k'^\mu + k^\mu) + i\sigma^{\mu\nu}(k'_\nu - k_\nu)] U_\pm(\vec{k}), \quad (4.22a)$$

whereas for negative-energy solutions

$$\bar{V}_\pm(\vec{k}') \gamma^5 \gamma^\mu V_\pm(\vec{k}) = \frac{1}{2m} \bar{V}_\pm(\vec{k}') [(k'^\mu + k^\mu) + i\sigma^{\mu\nu}(k'_\nu - k_\nu)] V_\pm(\vec{k}). \quad (4.22b)$$

For  $k' = k$ , this simplifies to

$$\bar{U}_\pm(\vec{k}) \gamma^5 \gamma^\mu U_\pm(\vec{k}) = -\frac{1}{m} \bar{U}_\pm(\vec{k}) k^\mu U_\pm(\vec{k}), \quad (4.23a)$$

$$\bar{V}_\pm(\vec{k}) \gamma^5 \gamma^\mu V_\pm(\vec{k}) = \frac{1}{m} \bar{V}_\pm(\vec{k}) k^\mu V_\pm(\vec{k}). \quad (4.23b)$$

The results (4.22) and (4.23) for the tachyonic *axial vector* current have a similar structure as the Gordon decomposition for the tardyonic *vector* current obtained with the ordinary Dirac equation [see Eq. (2.54) of Ref. [59]]. The role of the Dirac adjoint for the tardyonic case is taken over by the “chiral adjoint”  $\bar{U}_\pm(\vec{k}) \gamma^5$  for the tachyonic particle. Here, the designation “chiral adjoint” is inspired by the fact that  $\bar{U}_\pm(\vec{k}) \gamma^5 U_\pm(\vec{k})$  transforms as a pseudo-scalar under Lorentz transformations.

The structure of Eqs. (4.21) and (4.23) is somewhat peculiar with regard to parity. In Eq. (4.21a), an apparent vector current on the left-hand side appears to transform into an axial current on the right-hand side, whereas in Eq. (4.23a), an apparent axial vector on the left-hand side of the equation becomes what appears to be a vector on the right-hand side. The reason lies in the more complicated behavior of the tachyonic Dirac equation under parity as investigated in Ref. [10]. Namely, the tachyonic Dirac equation (1.7) contains a term which transforms as a scalar under parity,

$$i \gamma^\nu \partial_\mu \xrightarrow{\mathcal{P}} \gamma^0 (i \gamma^0 \partial_0 + i \gamma^i (-\partial_i)) \gamma^0 = i \gamma^\nu \partial_\mu \quad (4.24a)$$

as well as a term which transforms as a pseudoscalar,

$$\gamma^5 m \xrightarrow{\mathcal{P}} \gamma^0 (\gamma^5 m) \gamma^0 = -\gamma^5 m. \quad (4.24b)$$

The mass term in the tachyonic Dirac equation is pseudoscalar and changes sign under parity. Indeed, in Ref. [10], the tachyonic Dirac equation has been shown to be separately  $\mathcal{CP}$  invariant, and  $\mathcal{T}$  invariant, but not  $\mathcal{P}$  invariant, due to the change in the mass term.

In order to put this observation into perspective, we recall that the entries of the electromagnetic field strength tensor are composed of axial vector components (magnetic  $\vec{B}$  field), as well as vector components (electric  $\vec{E}$  field). The transformation properties of the electromagnetic field strength tensor under the proper orthochronous Lorentz group are nevertheless well-defined.

The transformation (4.24b) can be interpreted as a transformation  $m \rightarrow -m$  under parity. Thus, if we interpret the mass  $m$  as a pseudoscalar quantity, then the right-hand sides of (4.21a) and (4.23a) transform as a vector and an axial vector, respectively. It is the parity non-invariance of the mass term in the tachyonic Dirac equation which leads to the somewhat peculiar structure of Eqs. (4.21a) and (4.23a).

#### D. Helicity-Dependence and Gupta-Bleuler Condition

The anticommutator relations for tardyons given in Eq. (4.14) imply that both left-handed as well as right-handed helicity states, for both particles as well as antiparticles, have positive norm. However, the anticommutator relations for tachyons given in Eq. (4.15) imply that right-handed particle as well as left-handed antiparticle states acquire negative norm. This is shown in Eqs. (31) and (32) of Ref. [9]. Indeed, for one-particle states  $|1_{k,\sigma}\rangle = b_\sigma^\dagger(k)|0\rangle$ ,

$$\langle 1_{k,\sigma} | 1_{k,\sigma} \rangle = \langle 0 | b_\sigma(k) b_\sigma^\dagger(k) | 0 \rangle = \langle 0 | \{b_\sigma(k), b_\sigma^\dagger(k)\} | 0 \rangle = (-\sigma) V \frac{E}{m}, \quad (4.25)$$

where  $V = (2\pi)^3 \delta^3(\vec{0})$  is the normalization volume in coordinate space. The Fock-space norm  $\langle 1_{k,\sigma} | 1_{k,\sigma} \rangle$  is negative for  $\sigma = 1$ .

For clarification, the corresponding Gupta-Bleuler condition should be indicated explicitly. In full analogy to the instructive discussion of the Gupta-Bleuler mechanism for the photon field, as given in full clarity in Chap. 9b of Schweber's textbook [63], we select the positive- and negative-frequency component of the ( $\sigma = 1$ )-component of the neutrino field operator,

$$\psi_{\sigma=1}^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} b_{\sigma=1}(k) \mathcal{U}_{\sigma=1}(\vec{k}) e^{-ik \cdot x}, \quad (4.26a)$$

$$\psi_{\sigma=1}^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} d_{\sigma=1}^\dagger(k) \mathcal{V}_{\sigma=1}(\vec{k}) e^{ik \cdot x}, \quad (4.26b)$$

and postulate that it annihilates any physical Fock state  $|\Psi\rangle$  of the tachyonic field,

$$\langle \Psi | \psi_{\sigma=1}^{(-)}(x) = \psi_{\sigma=1}^{(+)}(x) | \Psi \rangle = 0. \quad (4.27)$$

These relations automatically imply that the Gupta-Bleuler condition also is realized in terms of the expectation value

$$\boxed{\langle \Psi | \psi_{\sigma=1}(x) | \Psi \rangle = \langle \Psi | \psi_{\sigma=1}^{(-)}(x) + \psi_{\sigma=1}^{(+)}(x) | \Psi \rangle = 0,} \quad (4.28)$$

but the condition (4.27) is stronger. We recall that the Gupta-Bleuler condition on the photon field reads  $\langle \Psi_\gamma | \partial^\mu A_\mu | \Psi_\gamma \rangle = 0$ , where  $|\Psi_\gamma\rangle$  is a Fock state of the photon field. As stressed in Schweber's book [63] on p. 246,

the condition  $\langle \Psi_\gamma | \partial_\mu A^\mu | \Psi_\gamma \rangle = 0$  is not sufficient for the suppression of the longitudinal and scalar photons, but one must postulate that  $\partial^\mu A_\mu^{(+)} | \Psi_\gamma \rangle = 0$ , where  $A_\mu^{(+)}$  is the positive-frequency component of the photon field operator. Because the tachyonic fermion, unlike the photon, is not equal to its own antiparticle, we need two conditions, given in Eq. (4.27).

A crucial question now concerns the possibility of reversing the helicity-dependence, i.e., the question of whether or not a different choice for the helicity-dependent factors in Eq. (4.3) exists that would imply negative norm for left-handed particles and right-handed antiparticles. A related question is whether other tachyonic Dirac Hamiltonians exist for which the choice  $f(\sigma, \vec{k}) = g(\sigma, \vec{k}) = +\sigma$  instead of  $(-\sigma)$  would fulfill our general postulate, namely, the sum rule (4.5). For reasons outlined in the following, we can ascertain that this is not the case; tachyonic spin-1/2 particles should always be left-handed.

The arguments supporting this conclusion are as follows. First, if we assume that the tachyonic field fulfills a sum rule of type II which for massless fields is given in Eq. (2.6), then it is impossible to replace  $(-\sigma)$  by  $(+\sigma)$  in the sum rule because of the necessity to preserve a smooth massless limit. The considerations in the text following Eq. (2.6) imply that if one were to replace  $(-\sigma)$  by  $(+\sigma)$  in the massless case, then one would violate the sum rule (2.6). The second argument is obtained by explicit calculation. We have checked that if one replaces  $m \rightarrow -m$  in the tachyonic and imaginary-mass Dirac equations (1.7) and (1.9), then the sum rules fulfilled by the corresponding fundamental spinors still contain the characteristic factor  $(-\sigma)$ . For the imaginary-mass Dirac equation, this result is obtained in Ref. [25]. Intuitively, we can understand this result as follows: The mass  $m$  in the denominators of the right-hand sides of Eqs. (3.7) and (3.12) is obtained as the modulus  $\sqrt{m^2} = |m|$  and does not change if we replace  $m \rightarrow -m$  in the superluminal Dirac equation. The mass in the numerator of the right-hand sides of Eqs. (3.7) and (3.12) changes sign, but this is consistent with the obvious change in the functional form of the positive-energy and negative-energy projectors as we change the sign of the mass term. Again, this consideration supports the conclusion that we cannot invert the helicity-dependence by choosing a different Hamiltonian; the factor  $(-\sigma)$  persists.

The third argument comes from the tachyonic Gordon identities discussed in Sec. IV C. We use the Gordon identity Eq. (4.23) and the normalization (3.6) to calculate the bispinor trace (with a  $\gamma^0$  multiplied from the right) of the left-hand side of Eq. (3.7a),

$$\begin{aligned} \text{tr} \left( \sum_{\sigma} (-\sigma) \mathcal{U}_{\sigma}(\vec{k}) \otimes \bar{\mathcal{U}}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 \right) &= \sum_{\sigma} (-\sigma) \bar{\mathcal{U}}_{\sigma}(\vec{k}) \gamma^5 \gamma^{\mu=0} \mathcal{U}_{\sigma}(\vec{k}) \\ &= \sum_{\sigma} (-\sigma) \left( -\frac{k^0}{m} \right) \underbrace{\bar{\mathcal{U}}_{\sigma}(\vec{k}) \mathcal{U}_{\sigma}(\vec{k})}_{=1} = 2 \frac{E}{m}. \end{aligned} \quad (4.29a)$$

The bispinor trace of the right-hand side of Eq. (3.7a) is

$$\text{tr} \left( \gamma^0 \frac{\not{k} - \gamma^5 m}{2m} \right) = \text{tr} \left( \gamma^0 \frac{\not{k}}{2m} \right) = 4 \frac{E}{2m} = 2 \frac{E}{m}, \quad (4.29b)$$

which shows the consistency of the bispinor sum (3.7a) with the Gordon decomposition (4.23). If we were to replace  $(-\sigma)$  by  $(+\sigma)$ , the two sides of the relation (4.29a) would differ by a minus sign. The bispinor trace of the left-hand side of Eq. (3.7b) is

$$\begin{aligned} \text{tr} \left( \sum_{\sigma} (-\sigma) \mathcal{V}_{\sigma}(\vec{k}) \otimes \bar{\mathcal{V}}_{\sigma}(\vec{k}) \gamma^5 \gamma^0 \right) &= \sum_{\sigma} (-\sigma) \bar{\mathcal{V}}_{\sigma}(\vec{k}) \gamma^5 \gamma^{\mu=0} \mathcal{V}_{\sigma}(\vec{k}) \\ &= \sum_{\sigma} (-\sigma) \left( \frac{k^0}{m} \right) \underbrace{\bar{\mathcal{V}}_{\sigma}(\vec{k}) \mathcal{V}_{\sigma}(\vec{k})}_{=-1} = 2 \frac{E}{m}. \end{aligned} \quad (4.30a)$$

We have used the tachyonic Gordon decomposition for negative-energy states as given in Eq. (4.23), which differs from the positive-energy Gordon decomposition by a minus sign, but an additional minus sign is obtained from the Lorentz-invariant normalization of the negative-energy fundamental bispinors. From the right-hand side of Eq. (3.7b), we have

$$\text{tr} \left( \gamma^0 \frac{\not{k} + \gamma^5 m}{2m} \right) = \text{tr} \left( \gamma^0 \frac{\not{k}}{2m} \right) = 2 \frac{E}{m}, \quad (4.30b)$$

which again is fully consistent, but only because we have a  $(-\sigma)$  in the sum rule (3.7), which combines with the  $(-\sigma)$  from the Lorentz-invariant normalization of the  $\bar{\mathcal{V}}_{\sigma}(\vec{k}) \mathcal{V}_{\sigma}(\vec{k})$ , to give  $2E/m$  as a final result in Eqs. (4.30a) and (4.30b).



## V. PHYSICAL INTERPRETATION: FROM TACHYONIC NEUTRINOS TO COSMOLOGY

### A. Arguments for and against Tachyonic Neutrinos

In the absence of conclusive experimental evidence, the hypothesis of tachyonic neutrinos has been controversially discussed in the literature. Three main arguments [64] have been brought forward against tachyonic neutrinos. **(1.)** They would require us to give up the notion of a Lorentz-invariant vacuum state, and even the vacuum would become unstable in the presence of tachyonic fields. **(2.)** Given the tachyonic dispersion relation  $E = \sqrt{\vec{k}^2 - m^2}$ , the role of states with  $|\vec{k}| < m$  needs to be clarified. **(3.)** The physical (probability!?) interpretation of the conserved Noether current of the free tachyonic Dirac equation has been called into question [64], and it has been argued that no consistent interpretation can be given because certain zero components of the conserved current were conjectured to vanish for all tachyonic momentum eigenstates [64].

A possible answer for question **(1.)** has been proposed in Ref. [9]. Summarizing the argument, it has been concluded in Ref. [9] that one can solve the problem in two ways. (i) One can Lorentz transform the vacuum state, and Lorentz transform all fundamental creation and annihilation operators of the fermion field (some of these will change from annihilators to creators upon transformation, due to the space-like nature of the tachyons). (ii) One keeps a Lorentz-invariant vacuum state, and only transform the space-time *arguments*  $k^\mu$  and  $x^\mu$  of the field operators, keeping all creation operators as creators and annihilation operators as annihilators. The amplitudes, cross sections, etc. obtained using approach (ii) then depend on scalar product of four-vectors which are equal to the result obtained by first calculating the process in the original Lorentz frame, and then, performing the Lorentz transformation into the moving frame.

The conjecture regarding an expansion about a “false” vacuum in the presence of tachyons can be traced to the fact that most of the tachyonic theories discussed so far in the literature are scalar [28–33]. Indeed, scalar tachyons have a problem with instability, because of the structure of the mass term in relation to the field Hamiltonians, which changes sign  $m^2 \rightarrow -m^2$  in a tachyonic theory, suggesting that the field energy can be lowered by creating tachyons. The problem does not occur in tachyonic spin-1/2 theories because a linear, not quadratic, mass term enters the field Lagrangian and Hamiltonian. Provided one reinterprets the spin-1/2 antiparticle solutions in the usual way (negative energy for propagation into the past becomes positive energy for propagation into the future), it then becomes immediately clear that the vacuum energy cannot be lowered upon spin-1/2 tachyon anti-tachyon pair production.

An solution to problem **(2.)** has also been proposed in Ref. [9]. Namely, the energies with  $E = \pm\sqrt{\vec{k}^2 - m^2} - i\epsilon$  (for  $|\vec{k}| < m$ ) find a natural interpretation in terms of complex resonance and antiresonance energies, which describe unstable states which decay in time. Particle resonances are damped for propagation into the future, antiparticle antiresonances are damped for propagation into the past, as they should be [9]. The occurrence of momentum eigenstates with real energies, and resonances with complex resonance energies, is a well-known phenomenon all across physics (e.g., in atomic physics, the ground-state energy of the helium atom is strictly real, whereas auto-ionizing resonances in the three-particle system have a manifestly complex resonance energy).

Another “myth” which should be refuted concerns a conceivable “runaway reaction” where a moving tachyon releases an arbitrarily large amount of energy, as it loses energy and accelerates, given its classical energy-velocity relation  $E = m/\sqrt{v^2 - 1}$ . According to this relation, a tachyon indeed accelerates as its energy is lowered and becomes commensurate with the invariant mass square. (At high energy, a tachyon approaches the light cone, though.) An infinitely fast tachyon takes the role of a tardyon at rest [28]; the explicit eigenstates have been indicated in Ref. [9]. However, energy conservation holds, and one cannot mix the notion of energy increase by acceleration, which only holds for tardyonic particles, in order to “convert” a tachyon losing energy by acceleration into a tardyon that gains energy in the same process. The “runaway reaction” is impossible; only a finite amount of energy is released as the tachyon accelerates and the energy goes from  $E = m/\sqrt{v^2 - 1}$  to zero. Energy conservation holds for tachyons, even if they accelerate, somewhat counterintuitively, when losing energy.

An answer to question **(3.)** has not yet been provided so far in the literature to the best of our knowledge. Here, we aim to provide a possible physical interpretation for the conserved current and scalar product, and also show where certain arguments presented originally by Hughes and Stephenson in their research article [64] entitled “against tachyonic neutrinos” become inconsistent. The Lagrangian density of the tachyonic Dirac particle reads, in first quantization [6],

$$\mathcal{L}(x) = \bar{\psi}(x) \gamma^5 (i\gamma^\mu \partial_\mu - \gamma^5 m) \psi(x). \quad (5.1)$$

This is equivalent, up to partial integration, to the symmetric form [10]

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^5 \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^5 \gamma^\mu \psi) - m \bar{\psi} \psi, \quad (5.2)$$

where we suppress the space-time argument  $x = (t, \vec{r})$ . The non-symmetric form (5.1) clearly exhibits the presence of the “chiral adjoint”  $\bar{\psi}(x)\gamma^5$  in the Lagrangian. The conserved current is

$$\mathcal{J}^\mu(x) = \bar{\psi}(x)\gamma^5\gamma^\mu\psi(x), \quad \partial_\mu\mathcal{J}^\mu(x) = 0. \quad (5.3)$$

The zero component

$$\mathcal{J}^0(x) = \bar{\psi}(x)\gamma^5\gamma^0\psi(x) \quad (5.4)$$

assumes the following values in plane-wave eigenstates, according to Eqs. (4.11a) and (4.11b),

$$\bar{\mathcal{U}}_\sigma(\vec{k})\gamma^5\gamma^0\mathcal{U}_\sigma(\vec{k}) = -\sigma\frac{E}{m}, \quad (5.5a)$$

$$\bar{\mathcal{V}}_\sigma(\vec{k})\gamma^5\gamma^0\mathcal{V}_\sigma(\vec{k}) = -\sigma\frac{E}{m}. \quad (5.5b)$$

We conclude that  $\mathcal{J}^0$  is positive for left-handed particle states ( $\sigma = -1$ ) and right-handed antiparticle states (likewise,  $\sigma = -1$ ; the helicity is equal to  $-\sigma$  for antiparticles). For  $E = 0$ , the “axial norm” of the momentum eigenstates given in Eq. (5.5) vanishes, but this is a very special case. Namely, for tachyons,  $E = 0$  implies  $|\vec{k}| = m = mv/\sqrt{v^2 - 1}$  for the momentum and thus corresponds to an “infinitely fast” tachyon ( $v \rightarrow \infty$ ), which remains “infinitely fast” upon Lorentz transformation. It would thus be wrong to conclude, as done in Ref. [64] in the text following Eq. (54) of Ref. [64], that all tachyonic states have zero axial norm, because this would correspond to a forbidden generalization of a result which holds asymptotically, for  $v \rightarrow \infty$ , to all finite values of the tachyonic velocity  $v$ . Our Eq. (5.5) gives the explicit result for any finite value of the energy  $E$ .

The spatial integral of the zero component of the conserved current,

$$\int d^3r \mathcal{J}^0(x) = \int d^3r \bar{\psi}(x)\gamma^5\gamma^0\psi(x) = - \int d^3r \psi^\dagger(x)\gamma^5\psi(x), \quad (5.6)$$

is precisely equal (up to a sign) to the scalar product  $\langle\psi_1(t), \psi_2(t)\rangle \equiv \int d^3r \psi_1^\dagger(t, \vec{r})\gamma^5\psi_2(t, \vec{r})$  which is conserved under the time-evolution by the  $\gamma^5$  Hermitian (pseudo-Hermitian) Hamiltonian  $H_5$ . This scalar product is not positive definite, as already noticed in the work of Pauli [13], and precisely corresponds to the scalar product introduced by Pauli in Eq. (3) of Ref. [13], where in the notation of Ref. [13] we have  $\eta = \gamma^5$ . Similar observations have been made in Eqs. (34) and (52) of Ref. [64]. According to Refs. [65, 66], one could otherwise define a so-called  $\mathcal{C}$  operator, which is not equal to the charge conjugation operator, and “remedies” the problem of negative norm attained by some states under the  $\gamma^5$  norm, leading to a redefined, positive-definite scalar product. However, the negative norm finds a rather natural interpretation, and a redefinition of the scalar product  $\langle\psi_1(t), \psi_2(t)\rangle \equiv \int d^3r \psi_1^\dagger(t, \vec{r})\gamma^5\psi_2(t, \vec{r})$  therefore is not required.

For the ordinary Dirac equation, the conserved current is  $J^\mu(x) = \bar{\psi}_e(x)\gamma^\mu\psi_e(x)$ . Its timelike component is  $J^0(x) = \psi_e^\dagger(x)\psi_e(x)$ , where the subscript  $e$  reminds us of the electron. The latter can be interpreted as a positive-definite probability density which is conserved under the time evolution generated by the ordinary Hermitian Dirac Hamiltonian  $H^{(1)}$ .

The tachyonic Dirac current  $\mathcal{J}^\mu$  is obtained from the ordinary Dirac  $J^\mu$  by the replacement  $\bar{\psi}_e(x) \rightarrow \bar{\psi}(x)\gamma^5$ . We have seen that the scalar product for the tachyonic Dirac Hamiltonian is equal to an integral over the timelike component of the conserved Noether current of the Dirac equation and is not positive-definite. In order to put this observation into perspective, it is instructive to recall that for the Klein-Gordon equation  $(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$ , the zero component of the conserved current  $j^\mu(x) = \frac{i}{2m}(\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x))$  is not positive-definite, either. Therefore, the zero component of the Klein-Gordon current cannot be interpreted as a probability density but must be interpreted as a charge density, which is positive for particles and negative for antiparticles.

This interpretation is not available for the zero component of the Noether current of the tachyonic Dirac equation, because the equation is not charge conjugation invariant and is primarily proposed to describe neutrinos [10]. However, we can come closer to a physical interpretation of the timelike component of the Noether current of the tachyonic equation if we compare the interaction Lagrangian  $\mathcal{L}_{\text{QED}}$  of quantum electrodynamics to the weak interaction  $\mathcal{L}_W$  of a neutrino and a “heavy photon”, i.e., a  $Z^0$  boson,

$$\mathcal{L}_{\text{QED}} = -e\bar{\psi}_e\gamma^\mu\psi_e A_\mu \quad \leftrightarrow \quad \mathcal{L}_W = -\frac{e}{2\sin\theta_W\cos\theta_W}\bar{\psi}\left(\gamma^\mu\frac{1-\gamma^5}{2}\right)\psi Z_\mu, \quad (5.7)$$

where  $\theta_W$  is the Weinberg angle. The axial vector part  $\mathcal{L}_W^A$  of  $\mathcal{L}_W$  is

$$\mathcal{L}_W^A = -\frac{e}{2\sin(2\theta_W)}\bar{\psi}\gamma^5\gamma^\mu\psi Z_\mu, \quad (5.8)$$

where the ordering of the  $\gamma$  matrices is important. For the interaction with the time-like component of the vector potential, we have the expressions

$$\mathcal{L}_{\text{QED}}^0 = -e \bar{\psi}_e \gamma^0 \psi_e A_0 \quad \leftrightarrow \quad \mathcal{L}_W^{A,0} = -\frac{e}{2 \sin(2\theta_W)} \bar{\psi} \gamma^5 \gamma^0 \psi Z_0. \quad (5.9)$$

For quantum electrodynamics (QED), we interpret  $\bar{\psi}_e \gamma^0 \psi_e = \psi_e^\dagger \psi_e$  as the probability density, and this suggests an interpretation of the expression  $\bar{\psi} \gamma^5 \gamma^0 \psi$  as an “axial probability density” or “axial interaction density” of the neutrino field with the timelike component of the  $Z^0$  boson. According to Eqs. (5.5a) and (5.5b), the “axial interaction density” is positive for the physically allowed states (left-handed particle and right-handed antiparticle states), and negative for the physically forbidden states (right-handed particle and left-handed antiparticle states). This consideration is independent of the suppression mechanism for the states of “wrong” helicity which, in second quantization, due to negative norm [see the discussion following Eq. (4.25)].

In general, the physical interpretation of tachyonic theories has been discussed by Feinberg [32, 33] and Sudarshan *et al.* [28–31]. The reinterpretation principle is a cornerstone of the theory. The only physically sensible quantities in a quantum theory are transition amplitudes. If the time-ordering of superluminal events changes upon Lorentz transformation, then one reinterprets the amplitude as connecting two space-time events whose coordinates are transformed according to the Lorentz transformation, so that the only physically sensible quantity (transition amplitude) simply connects two events one of which happens before the other [31]. This may seem counter-intuitive at first, but it is perhaps a little less counter-intuitive if we take into account that the accepted formulation of quantum field theory is based on the Feynman propagator and on the reinterpretation principle for the “advanced” part of the Feynman Green function, which propagates anti-particle solutions into the past. In order to avoid problems with regard to causality, the canonical quantum field theory of subluminal particles has to be supplemented by a reinterpretation principle, just like the tachyonic theory. The physical interpretation of the Noether current is not affected by this consideration.

Finally, let us point out that even without invoking reinterpretation, superluminal propagation can be compatible with causality if we postulate that the tachyonic mass  $m$  is so small that the superluminality is within the limits set forth by the uncertainty relation. With an energy  $E = m/\sqrt{\beta^2 - 1} \approx m/\sqrt{2\delta}$  with  $v = 1 + \delta$  and  $\delta \ll 1$ , we have  $\Delta E \Delta t \approx m/\sqrt{2\delta} \Delta t \leq 1$  (in units with  $\hbar = 1$ ). Thus, for an infinitesimal tachyonic mass parameter which does not exceed  $m \leq \sqrt{2\delta}/\Delta t$ , causality is preserved within the limits set by the uncertainty principle. Quantum limitations, the role of unstable modes and quantum tunneling in superluminal propagation have been discussed in Refs. [67–70].

## B. Gupta–Bleuler Condition and Seesaw Mechanism

The commonly accepted mechanism for the suppression of right-handed neutrino and left-handed antineutrino states is the seesaw mechanism [71]. After integrating the heavy degrees of freedom (the sterile neutrino), it contains a nonrenormalizable dimension-five operator and has a hierarchy problem: Namely, the neutrino masses are inversely proportional to the grand unification (GUT) scale  $\Lambda$  and sensitively depend on fine-tuning of  $\Lambda$ . Let us consider Eq. (18) of Ref. [71] and *hypothetically* consider a formulation that would result if the physically observable neutrino were right-handed. Then, we could reformulate Eq. (18) of Ref. [71] as

$$\mathcal{L}_I^{\text{eff}} = -\frac{1}{\Lambda} \sum_{r,r'} \left[ \bar{R}_{r'R} \tilde{H} \right] Y_{r'r} \left[ \tilde{H}^T (R_{rR})^c \right] + \text{h.c.}, \quad (5.10)$$

where

$$R_{rR} = \begin{pmatrix} \nu_{rR} \\ r_R \end{pmatrix}, \quad H = \begin{pmatrix} H^{(+)} \\ H^{(0)} \end{pmatrix}. \quad (5.11)$$

With the expectation value of the Higgs field,

$$\tilde{H}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad (5.12)$$

instead of the left-handed Majorana neutrino masses, the right-handed ones would be small,

$$\mathcal{L}^M = -\frac{1}{2} \sum_{r,r'} \bar{\nu}_{r'R} M_{rr'}^R (\nu_{rR})^c + \text{h.c.}, \quad M_{rr'}^R = \frac{v^2}{\Lambda} Y_{r'r}. \quad (5.13)$$

The seesaw mechanism is not unique in suppressing a definite helicity of the neutrino. It is unique provided one formulates it in terms of the left-handed fermion fields, but it could be formulated with inverted helicities if the helicity of the observed neutrinos at low energy were different. In the latter case, the right-handed neutrino mass would be small, instead of the left-handed one. On the other hand, the seesaw mechanism has the distinct advantage that it is not necessary to assume a superluminal character of the neutrino.

The mechanism discussed here in the text following Eq. (4.25) is definite in making a prediction regarding the suppression of right-handed neutrino states, as explained by three independent arguments in Sec. IV D. However, we have to assume a superluminal neutrino. While an interacting superluminal field theory is problematic, it is perhaps not as problematic as previously thought (see Ref. [72] and Sec. 4 of Ref. [9]).

Final clarification can only come from experiment. The seesaw mechanism is compatible with a Majorana neutrino. The tachyonic Dirac equation implies that the neutrino cannot be equal to its antiparticle, because it does not allow charge-conjugation invariant solutions. It is only  $\mathcal{CP}$ , but not  $\mathcal{C}$  invariant. Experimental evidence for neutrinoless double beta decay is disputed [73], and direct measurements of the neutrino mass square currently exclude neither positive nor negative values [44–50]. The generally accepted seesaw mechanism implies that neutrino masses are generated by a nonrenormalizable interaction with a concomitant hierarchy problem, and the mechanism in itself could be reformulated with opposite helicities. It is compatible with a Majorana neutrino. By contrast, a tachyonic neutrino is “automatically” left-handed and not equal to its own antiparticle. It is described by a  $\gamma^5$  Hermitian Hamiltonian and is plagued with the conceptual difficulties associated with (ever so slightly) superluminal propagation. This means that it is experimentally possible to test the models.

### C. Tachyonic Neutrinos as a Candidate for Dark Energy

The formulation of a gravitational interaction of a spin-1/2 particle is nontrivial in the quantized formalism. Brill and Wheeler [74] performed the pioneering steps in this direction. In order to formulate the gravitational coupling of a Dirac particle, one has to formulate generalized Dirac matrices  $\bar{\gamma}^\mu$ , which fulfill anticommutation relations compatible with the local metric  $\bar{g}^{\mu\nu}(x)$  of curved space-time. Based on the Christoffel symbols  $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(x)$ , one formulates the Christoffel affine connection matrices  $\Gamma_\mu$  in spinor space and calculates the covariant derivative  $\nabla_\mu$  as follows [74–78],

$$\{\bar{\gamma}^\mu(x), \bar{\gamma}^\nu(x)\} = 2\bar{g}^{\mu\nu}(x), \quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} \bar{g}^{\rho\sigma} \left( \frac{\partial \bar{g}_{\nu\sigma}}{\partial x^\mu} + \frac{\partial \bar{g}_{\mu\sigma}}{\partial x^\nu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\sigma} \right), \quad (5.14)$$

$$\nabla_\nu \bar{\gamma}_\mu = \frac{\partial \bar{\gamma}_\mu}{\partial x^\nu} - \Gamma^\rho_{\mu\nu} \bar{\gamma}_\rho + \bar{\gamma}_\mu \Gamma_\nu - \Gamma_\nu \bar{\gamma}_\mu = 0, \quad \Gamma_\mu = -\frac{1}{4} \bar{\gamma}^\nu \left( \frac{\partial \bar{\gamma}_\mu}{\partial x^\nu} - \bar{\gamma}_\sigma \Gamma^\sigma_{\nu\mu} \right). \quad (5.15)$$

The above formula for  $\Gamma_\mu$  is valid in the case of a diagonal metric  $\bar{g}^{\mu\nu}$  such as the Schwarzschild metric considered in Ref. [76]. For a general space-time metric, and with “West-Coast” conventions for the local vierbein  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  (“East-Coast” conventions were used in Ref. [74]), the result reads as

$$\Gamma_k = -\frac{i}{4} \bar{g}_{\mu\alpha} \left( \frac{\partial b_\nu^\beta}{\partial x^k} a^\alpha_\beta - \Gamma^\alpha_{\nu k} \right) \bar{\sigma}^{\mu\nu}, \quad (5.16)$$

where  $\bar{\sigma}^{\mu\nu} = \frac{i}{2} [\bar{\gamma}^\mu, \bar{\gamma}^\nu]$  is the spin matrix. The  $a^\alpha_\beta$  and  $b_\nu^\beta$  coefficients transform the Dirac matrices to the local vierbein,

$$\bar{\gamma}_\rho = b_\rho^\alpha \gamma_\alpha, \quad \gamma_\rho = a^\alpha_\rho \bar{\gamma}_\alpha, \quad \bar{\gamma}^\alpha = a^\alpha_\rho \gamma^\rho, \quad \gamma^\alpha = b_\rho^\alpha \bar{\gamma}^\rho. \quad (5.17)$$

With the covariant derivative  $\nabla_\mu = \partial_\mu - \Gamma_\mu$ , the gravitationally coupled Dirac equation reads as [74–76]

$$(i \bar{\gamma}^\mu \nabla_\mu - m) \psi(x) = 0. \quad (5.18)$$

In the case of a tachyonic Dirac particle, it has to be reformulated as follows,

$$\boxed{(i \bar{\gamma}^\mu \nabla_\mu - \bar{\gamma}^5(x) m) \psi(x) = 0.} \quad (5.19)$$

The space-time coordinate-dependent matrix  $\bar{\gamma}^5(x)$  can be defined as

$$\bar{\gamma}^5(x) = \frac{i}{4!} \frac{\bar{\epsilon}_{\alpha\beta\gamma\delta}}{\sqrt{-\bar{g}}} \bar{\gamma}^\alpha \bar{\gamma}^\beta \bar{\gamma}^\gamma \bar{\gamma}^\delta = \frac{i}{4!} \epsilon_{\alpha\beta\gamma\delta} \bar{\gamma}^\alpha \bar{\gamma}^\beta \bar{\gamma}^\gamma \bar{\gamma}^\delta = i \bar{\gamma}^0 \bar{\gamma}^1 \bar{\gamma}^2 \bar{\gamma}^3, \quad (5.20)$$

where  $\bar{g} = \det \bar{g}^{\mu\nu}$  is the determinant of the metric, and  $\bar{\varepsilon}_{\alpha\beta\gamma\delta} = \sqrt{-\bar{g}} \varepsilon_{\alpha\beta\gamma\delta}$  is the local  $\epsilon$  tensor, while  $\varepsilon_{\alpha\beta\gamma\delta}$  is the totally antisymmetric Levi-Civita tensor. The last identity in Eq. (5.20) is valid for a diagonal metric  $\bar{g}^{\mu\nu}$ .

Let us first discuss Eq. (5.18) very briefly. A projection onto the upper and lower radial components  $f(r)$  and  $g(r)$  in a gravitational field can be found in Eqs. (19) and (20) of Ref. [76]. For vanishing electrostatic potential  $V \rightarrow 0$ , Eq. (20) of Ref. [76] is invariant under the replacement  $f(r) \leftrightarrow g(r)$ , and  $E \leftrightarrow -E$ . So, if  $E$  is an eigenvalue of the gravitationally coupled Dirac equation, so is  $-E$ . Invoking reinterpretation and replacing  $-E \rightarrow E$  for antiparticles, we find that the spectrum of the gravitationally coupled Dirac Hamiltonian is the same for particles and antiparticles. Therefore, the formalism makes the unique prediction that tardyonic antiparticles, like tardyonic particles, are attracted by a gravitational field. In passing, we note that the often cited motivation for the investigation of trapped antihydrogen and its interaction with the gravitational field therefore is faced with a unique theoretical prediction: Antiparticles are attracted by gravitation as much as particles are.

It has been confirmed within the last two decades [79–82] that the Universe expands more rapidly on large distance scales than compatible with the matter density in the Universe. Coupling to a scalar field (“quintessence”) is usually invoked in order to explain the expansion of the expansion rate of the Universe [83]. As the scalar quintessence field “rolls down its potential”, it accelerates the expansion rate of the Universe. Because of a self-attracting property (the quintessence field energy can be lowered by increasing the local density of the quintessence field), quintessence has positive energy density but negative pressure. This property is necessary in order to constitute a candidate for dark energy [83]. In the standard model of cosmology, dark energy accounts for about 73% of the total mass-energy of the Universe. The quintessence field thereby acts like a time-dependent cosmological constant.

We can thus conclude that most of the energy in the Universe actually is not gravitationally attractive, i.e., that gravity can repel. In the following, we shall present qualitative arguments which suggest that tachyonic neutrinos may play a role in the expansion of the Universe, and may contribute to “dark energy”. Conceivable connections of tachyonic physics and dark energy have been explored in the literature, employing either scalar fields [84] or Lorentz-violating mechanisms [85]. In order to provide an alternative explanation for dark energy, it is necessary to invoke a mechanism that leads to a repulsive gravitational force on intergalactic distance scales in the Universe. It is not fully surprising that tachyonic neutrinos may provide for such an alternative mechanism. Namely, both on the classical level [86], as well as on the level of quantum theory [Eq. (5.19)], tachyonic particles are repulsed by gravitational fields.

On the classical level, this is seen as follows. We start from the familiar equation of motion of a particle with mass  $m$  in curved space-time [see Eq. (86') of Ref. [86]], which is a geodesic,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0. \quad (5.21)$$

With a suitably redefined proper time  $ds'$ , the zero geodesic for a tachyon reads as [see Eq. (86') of Ref. [86]]

$$\frac{d^2 x'^\mu}{ds'^2} + \Gamma'^\mu_{\rho\sigma} \frac{dx'^\rho}{ds'} \frac{dx'^\sigma}{ds'} = 0. \quad (5.22)$$

The force exerted on a tardyon reads, according to Eq. (79a) of Ref. [86],

$$F^\mu = m \frac{d^2 x^\mu}{ds^2} = -m \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}. \quad (5.23)$$

However, there is a sign change for a tachyon [see Eq. (79b) of Ref. [86]],

$$F'^\mu = -m \frac{d^2 x'^\mu}{ds'^2} = m \Gamma'^\mu_{\rho\sigma} \frac{dx'^\rho}{ds'} \frac{dx'^\sigma}{ds'}, \quad (5.24)$$

corresponding to the change in the energy-velocity relation  $E = m/\sqrt{1-v^2} \rightarrow E = m/\sqrt{v^2-1}$  for tardyon versus tachyon, which leads to gravitational repulsion for tachyons. Another intuitive way to understand gravitational tachyonic repulsion is to observe that the quantity  $E = \sqrt{\vec{k}^2 + m^2}$  gets smaller when the position-dependent mass  $m = m(x)$  gets smaller, whereas the tachyonic energy  $E = \sqrt{\vec{k}^2 - m^2}$  gets larger when the position-dependent mass  $m = m(x)$  gets smaller.

As neutrinos are ubiquitous within the cosmos, one now needs to evaluate their conceivable contribution to the repulsive force on intergalactic distance scales. A detailed discussion is beyond the scope of the current article, but we can formulate some initial considerations and order-of-magnitude estimates. One can identify, a priori, two sources of neutrinos which may become important, (i) thermalized neutrinos which decouple in the early Universe [87–90], and (ii) non-thermalized, high-energy neutrinos which are continuously generated through various cosmic processes, such as nuclear fusion in stars like the sun, supernovae, and high-energy cosmic background.

It is generally assumed that today's cosmic background (CMB) radiation is accompanied by a neutrino background, which at some point (low energy) decoupled from the other particles. The average energy of the background neutrinos is generally assumed to be of the same order-of-magnitude as compared to the cosmic microwave background (CMB), and the currently accepted value for the temperature of the neutrino background is  $(4/11)^{1/3} = 0.714$  times that of the electromagnetic background radiation, i.e., about 1.9 K. The Universe, according to this hypothesis, would be filled with a sea of nonrelativistic background neutrinos. This picture changes when we assume that the neutrino is tachyonic. In the earliest stages of the Universe, tachyonic neutrinos are of course highly energetic, with  $E = \sqrt{\vec{k}^2 - m^2} \approx |\vec{k}| \gg 0$ . As they lose energy, they become faster, because  $E = m/\sqrt{v^2 - 1} \rightarrow 0$  (in the classical theory). The energy  $E = \sqrt{\vec{k}^2 - m^2}$  vanishes as  $|\vec{k}| \rightarrow m$ , within quantum theory. For  $|\vec{k}| < m$ , according to [9], the spectrum of the tachyonic Hamiltonian contains unstable resonances and anti-resonances, with a purely imaginary resonance energy. Because the real part of the resonance energy  $E = -i\sqrt{m^2 - \vec{k}^2}$  vanishes, the unstable states form a “background” of fluctuating quantum states in the Universe. Particle resonances ( $E = -i\sqrt{m^2 - \vec{k}^2}$ ) and antiparticle antiresonances ( $E = +i\sqrt{m^2 - \vec{k}^2}$ ) are damped for propagation into the future and past, respectively (see Ref. [9]). The latter case amounts to propagation into the future if one invokes reinterpretation for the antiparticle solutions [9].

A very coarse-grained estimate regarding the role of the fluctuating resonance and antiresonance states can be given as follows. We start from the grand canonical ensemble

$$\Omega = -2 \frac{k_B T}{2\pi^2} V \int_0^\infty dk k^2 \ln \left( 1 + \exp \left( \frac{\mu - E(k)}{k_B T} \right) \right), \quad (5.25)$$

where  $k_B$  is the Boltzmann constant,  $V$  is the normalization volume,  $\mu$  is the chemical potential, and  $E(k)$  is the energy of a tachyon as a function of  $k = |\vec{k}|$ . The multiplicity factor 2 takes into account the particle and antiparticle resonances and is supplemented here in comparison to the derivation presented in Ref. [91]. Furthermore, we have assumed an isotropic energy dependence  $\int d^3k \rightarrow 4\pi \int dk k^2$ . The pressure is found as

$$p = -\frac{\Omega}{V} = \frac{k_B T}{\pi^2} \int_0^\infty dk k^2 \ln \left( 1 + \exp \left( \frac{\mu - E(k)}{k_B T} \right) \right). \quad (5.26)$$

For  $T \rightarrow 0$ , the contribution of resonances and antiresonances with  $|\vec{k}| < m$  to the pressure and energy density is easily evaluated according to Eqs. (27) and (28) of Ref. [91]. For the pressure, we have

$$p_0 = \frac{i}{3\pi^2} \int_0^m dk k^3 \frac{d\text{Im} E(k)}{dk} = \frac{i}{3\pi^2} \int_0^m dk \frac{k^4}{\sqrt{m^2 - k^2}} = \frac{i m^4}{16\pi}, \quad (5.27)$$

whereas the energy density is

$$\rho_0 = \frac{i}{\pi^2} \int_0^m dk k^2 \text{Im} E(k) = -\frac{i}{\pi^2} \int_0^m dk k^2 \sqrt{m^2 - k^2} = -\frac{i m^4}{16\pi} = -p_0. \quad (5.28)$$

The equation of state fulfilled by the zero-temperature limit of the tachyonic Dirac sea is  $w_0 = p_0/\rho_0 = -1$ . This is the required equation of state for a “vacuum” energy density that describes a nonvanishing cosmological constant [92]. If  $p_0 + \rho_0 = 0$ , then there is no net energy gain upon pulling on a “piston” which contains the Universe (an illustrative discussion can be found in Ref. [93]).

Both results in Eqs. (5.27) and (5.28) are imaginary. This raises the pertinent question of how to incorporate the imaginary pressure and energy density of the neutrino field into the evolution equations of the Universe. In giving rise to unstable resonance states, the tachyonic Dirac field, if it exists, would be different from any other known fundamental quantum fields [59]. An exhaustive mathematical description would require an extension of scattering theory to complex energy and momentum exchanges, applied to low-energy collisions with neutrinos, and describe the continuous decay and repopulation of the unstable tachyonic states in the Universe. A detailed discussion is beyond the scope of the current article. However, in a first approximation, we can argue that an established technique for the mathematical treatment of complex resonances entails complex scaling (for reviews, see Refs. [94, 95]), and we thus explore a complex scaling ansatz to the solution of the evolution equations in the following.

The imaginary results in Eqs. (5.27) and (5.28) describe the quantum fluctuations due to the unstable states. We thus consider Eqs. (8)–(11) of Ref. [92], which relate the accelerated expansion rate  $\ddot{a}/|a|$  and the Hubble constant  $H = \dot{a}/|a|$  (the dot denotes differentiation with respect to time) to the energy densities of various cosmologically



relevant quantities,

$$H^2 = \left( \frac{\dot{a}}{|a|} \right)^2 = \frac{8\pi G}{3} (\rho_\Lambda + \rho_M + \rho_k) \approx \frac{8\pi G}{3} (\rho_\Lambda + \rho_M) , \quad (5.29a)$$

$$\frac{\ddot{a}}{|a|} = \frac{8\pi G}{3} \rho_\Lambda - \frac{4\pi G}{3} (\rho_M + 3p_M + \rho_k + 3p_k) \approx \frac{4\pi G}{3} (2\rho_\Lambda - \rho_M) . \quad (5.29b)$$

Here,  $\rho$  is the matter density in the Universe,  $\rho_k = -3k/(8\pi G a^2)$  is the energy density associated with the curvature of the Universe ( $k = +1, 0, -1$ ), while  $\rho_\Lambda$  is the energy density corresponding to the cosmological constant. The mass density is  $\rho_M$  and  $p_M \approx 0$ . We aim to solve Eq. (5.29) using a complex variable ansatz  $a = |a| \exp(i\theta)$  and  $t = t \exp(i\varphi)$ . Using Eq. (5.29), with a value  $H = 71 \text{ km s}^{-1} \text{ Mpc}^{-1}$  for the Hubble constant, one easily reproduces the accepted value for the critical mass density of  $\rho_{\text{crit}} \approx 9.34 \times 10^{-33} \text{ kg cm}^{-3}$  (in SI units) which corresponds to a critical energy density of  $\rho_{\text{crit}} \approx 8.39 \times 10^{-9} \text{ J m}^{-3}$  (we have set  $c = 1$ ). In natural units, the result converts to  $\rho_{\text{crit}} \approx 9.99 \times 10^{-45} \text{ GeV}^4$ .

We courageously set  $k = 0$  in Eq. (5.29) (flat Universe) and match the modulus of the (complex) accelerated expansion  $\ddot{a}/|a|$  against the currently accepted values of  $\Omega_M = \rho/\rho_{\text{crit}} \approx 0.27$  (see Ref. [96]) and  $\Omega_\Lambda = \rho_\Lambda/\rho_{\text{crit}} \approx 0.73$ . According to Eq. (5.29b), the matching is performed by courageously assuming that the bulk of the cosmological constant is given by the complex energy density (5.28), and equating the complex modulus as follows,

$$\rho'_\Lambda \approx \rho_0 = -\frac{i m^4}{16\pi} , \quad |2\rho'_\Lambda - \rho_M| = \sqrt{4|\rho'_\Lambda|^2 + \rho_M^2} \stackrel{!}{=} 2\rho_\Lambda - \rho_M \approx 1.19 \rho_{\text{crit}} . \quad (5.30)$$

The solution is  $\rho'_\Lambda = -i0.579 \rho_{\text{crit}}$ , and so our estimate for the neutrino mass reads as

$$(5.79 \times 10^{-45} \text{ GeV}^4) \sim |\rho_0| = \frac{m^4}{16\pi} \quad \Rightarrow \quad \boxed{m \sim 0.0232 \text{ eV}} . \quad (5.31)$$

If this treatment is valid, then the mass  $m$  necessarily has to be the mass of the “heaviest” neutrino mass eigenstate, i.e., the one with the largest modulus of the tachyonic mass  $m$ . Heavier eigenstates would lead to a larger cosmological constant, because  $\rho_0$  is proportional to the fourth power of the neutrino mass. In terms of a conversion to flavor eigenstates, the heaviest mass eigenstates might well be the one closest to the electron neutrino and electron antineutrino lepton flavor eigenstates [97].

We have interpreted the quantities  $p_0$  and  $\rho_0$  given in Eqs. (5.27) and (5.28) as contributions to a cosmological constant which describes the evolution of the Universe in complex space and time directions  $a = |a| \exp(i\theta)$  and  $t = t \exp(i\varphi)$ . We should supplement the solutions for the complex rotation angles for space and time, which read as  $\theta = 38.1^\circ$  and  $\varphi = 70.6^\circ$ , within our complex scaling approach. The evolution of the energy density with the aging of the Universe is given by [see Eq. (12) of Ref. [92]]

$$\rho_0 = \rho_i(0) a^{-3(1+w_0)} = \text{const.} , \quad w_0 = -1 . \quad (5.32)$$

We therefore obtain a time-independent energy density  $\rho_0$ , which is consistent with the fundamental character of the fluctuating unstable resonances of the neutrino field. We reemphasize that the mass  $m \sim 0.0232 \text{ eV}$  is tachyonic and enters the gravitationally coupled tachyonic Dirac equation (5.19). The given mass value is not currently excluded by any terrestrial experiment, and it is consistent with the observed time spread of the arrival times of neutrinos from the supernova SN1987A (see Ref. [98]). The hypothesis that very light elementary spin-1/2 particles could be tachyonic has been pursued elsewhere [99].

## VI. CONCLUSIONS

In the current investigation, we present the fundamental solutions of generalized Dirac equations in the helicity basis, in a systematic and unified manner. Of particular importance are the Dirac equation with two tardyonic mass terms  $m_1 + i\gamma^5 m_2$  and two tachyonic mass terms  $im_1 + \gamma^5 m_2$ . Let us summarize the main results. We have discussed the ordinary (tardyonic) Dirac equation in Sec. IIB, a tardyonic Dirac equation (with two mass terms) in Sec. IIC, and two tachyonic Dirac equations in Secs. IIIA and IIIB. We give the fundamental eigenspinors that enter the plane-wave solutions of all of these equations [see Eqs. (2.10), (2.11), (2.17), (2.18), (3.2), (3.3), (3.9) and (3.10)]. The eigenspinors are obtained using projector techniques as outlined in Chap. 2 of Ref. [59]. For the “normal” Dirac equation [see Eqs. (2.10) and (2.11)], our results are consistent with Ref. [3] and Chap. 23 of Ref. [60]. For the



generalized Dirac equations, the solutions have not yet appeared in the literature in the compact form given in the current article, to the best of our knowledge. For the ordinary (tardyonic) Dirac equation, for the tachyonic Dirac equation, and for the imaginary-mass Dirac equation, the prefactors are brought into compact analytic form [see Eqs. (2.10), (2.11), (3.2), (3.3), (3.9) and (3.10)].

Finally, in Secs. IV A and IV B, we find that the tardyonic and tachyonic theories can be unified on the basis of the structurally simple anticommutator relations given in Eqs. (4.14) and (4.15), which are independent of the magnitude of the mass terms. As outlined in Sec. IV A, the coefficient functions  $f = f(\sigma, \vec{k})$  and  $g = g(\sigma, \vec{k})$  in the postulated form of the anticommutators (4.3) enter the tensor sums over the fundamental spinors in Eq. (4.5). For the egregiously simple choices indicated in Eqs. (4.14) and (4.15), which are consistent with a smooth massless limit, the tensor sums over the fundamental eigenspinors yield the positive-energy and negative-energy projectors, for both tardyonic as well as tachyonic eigenspinors. Consistency with the massless limit requires the presence of the factor  $(-\sigma)$  in the tensor sums over the eigenspinors for the tachyonic equations. This fact is verified, on the basis of tachyonic Gordon identities and related considerations, in Secs. IV C and IV D. The presence of the factor  $(-\sigma)$  in the fundamental tachyonic field anticommutators in Eq. (4.15) implies the suppression of right-handed particle and left-handed antiparticle states, due to negative norm, as shown in Eq. (4.25).

Finally, in Sec. V, we observe that since tachyons are repelled by gravity, it might be worthwhile to investigate their conceivable role in the mechanism(s) responsible for the accelerated expansion of the Universe (“dark energy”). Furthermore, the tachyonic resonances and anti-resonance energies might play a role in the sum over states that enters the thermodynamic potentials of a free tachyonic fermionic gas in the low-temperature limit. If we consider the states with imaginary energy to be unstable, fluctuating states, then it is intuitively obvious that they might contribute to a fluctuating energy density and pressure on large distance scales in the Universe. As described in a somewhat approach in Sec. V C, an order-of-magnitude calculation based on a complex scaling transformation of the time evolution of the Universe leads to a tachyonic neutrino mass which is not excluded at present by any terrestrial experiment. We believe that it might be worthwhile to explore the physical consequences of the “ugly duckling” (tachyonic neutrino) somewhat further. It allows us to retain, among other things, lepton number conservation as a symmetry of nature.

### Acknowledgments

The authors acknowledge helpful conversations with C. Hirata, R. J. Hill, W. Rodejohann, P. J. Mohr, and I. Nándori. This work was supported by the NSF (grant PHY-1068547) and by the National Institute of Standards and Technology (precision measurement grant).

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The slides are available from [http://regmedia.co.uk/2012/06/11/neutrinos\\_not\\_ftl\\_slides.pdf](http://regmedia.co.uk/2012/06/11/neutrinos_not_ftl_slides.pdf).

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What happens if Dr. Spock overtakes a left-handed neutrino...



Spinor Sum for Positive-Energy States:

$$\sum_{\sigma} (-\sigma) u_{\sigma}(\vec{k}) \otimes \bar{u}_{\sigma}(\vec{k}) \gamma^5 = \frac{\not{k} - \gamma^5 m}{2m}$$

Spinor Sum for Negative-Energy States:

$$\sum_{\sigma} (-\sigma) v_{\sigma}(\vec{k}) \otimes \bar{v}_{\sigma}(\vec{k}) \gamma^5 = \frac{\not{k} + \gamma^5 m}{2m}$$

*There is a paradox which results if one overtakes a left-handed neutrino, looks back and sees the same particle right-handed, because right-handed neutrinos have never been observed in nature. One possible way to solve the paradox is sketched in the current article; the spinor sums are given in Eq. (3.7). Further explanations are in the text.*